SKETCHES OF SOLUTIONS : ENDTERM EXAMINATION - ALGEBRA I - AUGUST-NOVEMBER 2024

Note: This document has not been checked for errors or inaccuracies and as such may contain some. That DOES NOT mean that the problems on the exam were necessarily wrong.

(1) Let A be the following matrix, considered as a matrix over complex numbers:

$$\begin{bmatrix} 1 & 0 & -4 \\ -6 & -1 & -12 \\ 0 & 0 & -1 \end{bmatrix}.$$

(i) Find eigenvalues of A.
(ii) Find a basis for each eigenspace corresponding to each eigenvalue of A.
(iii) Give an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$, or explain why none such can exist.

(10 marks)

Sketch of solution: Characteristic polynomial of A is determinant of $(xI - A)$. Expanding the determinant using the last row, we get the characteristic polynomial to be $(x-1)(x+1)^2$. So eigenvalue 1 appears with algebraic multiplicity 1 and eigenvalue -1 appears with algebraic multiplicity 2.

For a basis of the eigenspaces: For eigenvalue 1, solve the system of linear equations

$$\begin{bmatrix} 0 & 0 & -4 & | & 0 \\ -6 & -2 & -12 & | & 0 \\ 0 & 0 & -2 & | & 0 \end{bmatrix}.$$

Row reduce to get that the eigenspace is given by

$$x_2 \begin{bmatrix} -\frac{1}{3} \\ 1 \\ 0 \end{bmatrix}.$$

so a basis is given by

$$\begin{bmatrix} -\frac{1}{3} \\ 1 \\ 0 \end{bmatrix}.$$

For eigenvalue -1, solve the system of linear equations

$$\begin{bmatrix} 2 & 0 & -4 & | & 0 \\ -6 & 0 & -12 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}.$$

Row reduce to get that the eigenspace is given by

$$x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

so that a basis is given by

$$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

and the geometric multiplicity is 1.

Since algebraic multiplicity $\neq$ geometric multiplicity for each eigenvalue, we know that the matrix is not diagonalizable.

(2) Suppose A is an $n \times n$ matrix with real entries such that the diagonal elements are all positive, the off-diagonal elements are all negative, and the row sums are all positive. Prove that $\det(A) \neq 0$.
(10 marks)

Sketch of proof: It suffices to prove that the system of equations $AX = 0$ only admits the trivial solution. Let a solution be given by

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

Pick an index k such that $|x_k| = \max_{i=1,2,\ldots,n} |x_i|$. So that $|x_j| \leq |x_k|$ for all j. The idea is now to look at the kth row $(AX)_k$ of the equation $AX = 0$ and arrive at a contradiction.

If $x_k > 0$, then $x_j \leq |x_j| \leq |x_k| = x_k$ for all j. Then,

$$\begin{aligned} 0 = (AX)_k &= \sum_{j=1}^{n} A_{kj} x_j \\ &= A_{kk} x_k + \sum_{j \neq k} A_{kj} x_j \\ &\geq A_{kk} x_k + \sum_{j \neq k} A_{kj} x_k \text{ (since } A_{kj} < 0 \text{ for} j \neq k \text{ and } A_{kk} > 0.) \\ &= x_k \sum_{j=1}^{n} A_{kj} \\ &> 0, \end{aligned}$$

which is a contradiction.

The case $x_k < 0$ similarly leads to a contradiction once we make the appropriate changes of signs in the above equations.

So $x_k = 0$, but then $x_j = 0$ for all j by choice of x_k, so that X has to be the zero vector, and we are done.

Remark: This simple fact is called 'Minkowski's Criterion' and is quite useful.

(3) Find the shortest distance from the point $(1, 1, 2)$ to the plane $x_1 - x_2 + x_3 + 1 = 0$ in the 3-dimensional euclidean space with the usual dot product. Justify why your answer must be the shortest distance. (Note that this plane is NOT a vector subspace of the 3-dimensional euclidean space.)
(10 marks)

Sketch of solution: This follows from Gram-Schmidt process and orthogonal projections as we have seen in the class, after doing computations appropriately.

(4) This question is about symmetric bilinear forms.

(i) Let A and A' be symmetric matrices related by $A' = P^t AP$, where P is invertible. Show that the ranks of A' and A are equal. Show as a consequence that we can define the *rank* of a bilinear form $\langle , \rangle$ on a real finite dimensional vector space V as the rank of any matrix representing it.

(ii) Prove or disprove: a nonzero symmetric bilinear form $\langle , \rangle$ on a real finite dimensional vector space V is of rank 1 if and only if it is a product of two nonzero linear functionals, i.e. $\langle v, w \rangle = f_1(v)f_2(w)$ for $f_1, f_2 \in V^*$.

(15 marks)

Sketch of proof: For the first part, the easiest way is to note that P and P^t both correspond to bijective functions on image of A, so that they cannot make any change to the rank of A.

For the second part, in fact, there is no need to assume symmetric property of the bilinear form. Let us prove the result for any bilinear form $\langle , \rangle$. The below proof becomes much simpler if the bilinear form is assumed to be symmetric as in the original question.

Let $n = \dim(V)$. Consider two maps $f_L : V \to V^*$ defined by $f_L(x)(y) = \langle x, y \rangle$ and $f_R : V \to V^*$ defined by $f_R(x)(y) = \langle y, x \rangle$. Then, choose a basis $(v_1, v_2, \ldots, v_n)$ of V and the dual basis $(v_1^*, v_2^*, \ldots, v_n^*)$ of V^*. Then, by the definition of a dual basis, if we write matrices of these two maps in these bases, for one of them we get the matrix A associated to $\langle , \rangle$ and for the other we get its transpose A^t. Then we know that f_L and f_R both have rank 1 by hypothesis. So $\ker(f_L)$ has a basis of $n-1$ vectors, say $u_2, u_3, \ldots, u_n$. Similarly $\ker(f_R)$ has a basis of $n-1$ vectors, say $v_2, v_3, \ldots, v_n$.

Now let $u_1 \notin \ker(f_L)$ and $v_1 \notin \ker(f_R)$, and $c = \langle u_1, v_1 \rangle$. Then $c \neq 0$ (otherwise $\langle , \rangle$ is zero everywhere). Then, $u_1, u_2, \ldots, u_n$ and $v_1, v_2, \ldots, v_n$ are bases of V. Define $L_1 \in V^*$ by $L_1(u_1) = 1$ and $L_1(u_k) = 0$ for $k > 1$. Define also $L_2 \in V^*$ by $L_2(v_1) = c$ and $L_2(v_k) = 0$ for $k > 1$. Then $\langle u_i, v_j \rangle = L_1(u_i)L_2(v_j)$ for any i, j, and the result follows by bilinearity.

Remark: The first ever example of a bilinear map that you saw was the multiplication map $\mathbb{R} \times \mathbb{R} \to \mathbb{R} : (x, y) \to xy$. This is clearly rank 1. The above problem says that all rank 1 forms on any finite dimensional vector space essentially look like multiplication.

(5) Let V be a finite dimensional vector space over $\mathbb{C}$. Show that the signature of a Hermitian form $\langle , \rangle : V \times V \to \mathbb{C}$ is independent of the orthogonal basis chosen to write its matrix.
(15 marks)

Sketch of proof: This is Sylvester's Law, which was also on the homework. You can find the proof in many sources. In particular, one can prove that the signature is given by the number of positive eigenvalues and the number of negative eigenvalues, which is independent of the basis chosen.

(6) Let $n > 1$ be an integer. Let $\zeta_n = e^{\frac{2\pi\iota}{n}}$ and let A denote the $n \times n$ matrix whose entries are $a_{ij} = \frac{\zeta_n^{ij}}{\sqrt{n}}$. Is A a unitary matrix?
(15 marks)

Sketch of solution: Yes. This follows from taking the inner product of column vectors with themselves and summing appropriately over powers of ζ_n.

Remark: This simple technique and fact can be quite useful in representation theory and physics. Look up 'Complex Hadamard Matrices' for more information.

(7) Let $n > 1$ be an integer and A be a real $n \times n$ matrix, where all entries are zero except those on the diagonal and those in the first row and first column. Also assume that all diagonal entries of A are nonzero and that all entries in the first row and first column of A are strictly positive. Show that all the eigenvalues of A are real.
(10 marks)

Sketch of proof: The idea is to show that A is similar to a symmetric matrix. Let

$$A = \begin{bmatrix} a & \beta^t \\ \gamma & D \end{bmatrix}$$

where D is the $(n-1)\times(n-1)$ nonsingular diagonal matrix coming from the question, $a = a_{11}$ is the upper leftmost entry, and β and γ are vectors all whose entries, say $\beta_1, \dots, \beta_{n-1}$ and $\gamma_1, \dots, \gamma_{n-1}$ respectively, are positive.

Then, working with 2×2 or 3×3 matrices as an example, it is possible to arrive at a particularly simple way of symmetrizing. Namely, consider a diagonal matrix

$$P = \begin{bmatrix} 1 & \\ & Y \end{bmatrix}$$

where Y is an $(n-1)\times(n-1)$ diagonal matrix, say with diagonal entries $y_1, \dots, y_{n-1}$. Then define

$$B := PAP^{-1} = \begin{bmatrix} a & \beta^t Y^{-1} \\ Y\gamma & YDY^{-1} \end{bmatrix}.$$

Note that YDY^{-1} is a diagonal matrix, so that B is symmetric if and only if $\beta^t Y^{-1} = Y\gamma$. This happens if and only if

$$y_i \gamma_i = \frac{\beta_i}{y_i},$$

that is, if and only if

$$y_i^2 = \frac{\beta_i}{\gamma_i}$$

for each $i = 1, 2, \dots, n-1$. Since β_i and γ_i are strictly positive, we can choose such y_i and hence A is similar to B which is a symmetric matrix, and thus its eigenvalues are real.

(8) Let V be an infinite dimensional vector space over a field F. Let $T : V \to V$ be a linear operator such that $T(V)$, the image of T, is finite dimensional.

(i) Show that T satisfies a nonzero polynomial over F.

(ii) Assume, in addition, that $T^2(V) = T(V)$. Show that $V = \ker(T) \oplus T(V)$.

(15 marks)

Sketch of proof: For the first part, if the dimension of $T(V)$ equals n, choose a set $v_1, v_2, \dots, v_n$ of vectors in V such that $T(v_1), \dots, T(v_n)$ forms a basis for $T(V)$. Then, for each $i = 1, 2, \dots, n$, since $T(v_i), T^2(v_i), \dots, T^{n+1}(v_i)$ are all in $T(V)$, there has to be a nontrivial linear relation among them, and so for each i, there exists a polynomial $P_i(x)$ such that $P_i(T(v_i)) = 0$. Then take $P(x) = x\prod_{i=1}^n P_i(x)$ and see that T satisfies this polynomial (the first line of the next paragraph is used here, in fact).

For the second part, note that for any $v \in V$, $T(v) = T^2(x)$ for some x, so that $y = v - T(x)$ lies in kernel of T. Hence, $v = y + T(x)$ shows that $V = \ker(T) + T(V)$. To show that it is a direct sum, simply note that the linear operator $T_{\text{res}} : T(V) \to T(V)$ gotten by restriction of T to $T(V) \to T(V)$ is a surjection by hypothesis, and since $T(V)$ is finite dimensional, it also is injective by counting dimensions. So that kernel of T_{res} is $\{0\}$, which implies that it is a direct sum, since any element of $\ker(T) \cap T(V)$ is an element of this kernel.

Food for thought: Do eigenvalues for such operators appear as roots of some polynomial?

(9) This is an **EXTRA CREDIT** question. The solution to this question may depend on *significantly more difficult or different concepts* than what you may have seen during this course. So it is NOT recommended that you attempt and spend time on this question before attempting other questions.

Let V be an infinite dimensional vector space over $\mathbb{C}$ with basis $\mathcal{B}$.

(i) Show that $\mathcal{B}^* := \{v^* : v \in \mathcal{B}\}$ does not span V^*.
(ii) Show that V^* is isomorphic to the direct product of copies of $\mathbb{C}$ indexed by $\mathcal{B}$. Can you say something nontrivial about the dimension of V^* from this?

(20 marks)

Sketch of proof: Since this is an extra credit question, this is left as an exercise!