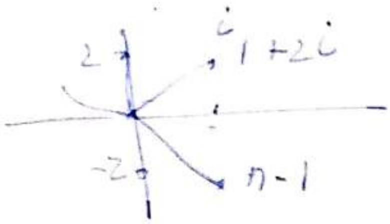

ENDTERM EXAMINATION - ALGEBRA I - AUGUST-NOVEMBER 2024

Instructions: The total time allotted for this examination is 3 hours. This is a closed book examination. Books, notebooks, cellphones, laptops and any such objects that may enable you to get external help are not allowed. Be brief but precise in your answers. You should justify all your assertions. In particular, if you want to use any problem from any homework, you cannot only quote it; you have to prove or solve it here. The maximum marks that can be scored on this exam are 120.

(1) Let A be the following matrix, considered as a matrix over complex numbers:

$$\begin{bmatrix} 1 & 0 & -4 \\ -6 & -1 & -12 \\ 0 & 0 & -1 \end{bmatrix}.$$

(i) Find eigenvalues of A. $1, -1$

(ii) Find a basis for each eigenspace corresponding to each eigenvalue of A. $\begin{pmatrix} 1 \\ -3 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

(iii) Give an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$, or explain why none such can exist. → No basis of eigen vectors.

(10 marks)

(2) Suppose A is an $n \times n$ matrix with real entries such that the diagonal elements are all positive, the off-diagonal elements are all negative, and the row sums are all positive. Prove that $\det(A) \neq 0$.
(10 marks)

(3) Find the shortest distance from the point $(1, 1, 2)$ to the plane $x_1 - x_2 + x_3 + 1 = 0$ in the 3-dimensional euclidean space with the usual dot product. Justify why your answer must be the shortest distance. (Note that this plane is NOT a vector subspace of the 3-dimensional euclidean space.)
(10 marks)

$\frac{\sqrt{6}}{3}$

(4) This question is about symmetric bilinear forms.

(i) Let A and A' be symmetric matrices related by $A' = P^tAP$, where P is invertible. Show that the ranks of A' and A are equal. Show as a consequence that we can define the *rank* of a bilinear form $\langle , \rangle$ on a real finite dimensional vector space V as the rank of any matrix representing it.

(ii) Prove or disprove: a non zero symmetric bilinear form $\langle , \rangle$ on a real finite dimensional vector space V is of rank 1 if and only if it is a product of two linear functionals, i.e. $\langle v, w \rangle = f_1(v)f_2(w)$ for $f_1, f_2 \in V^*$.

(15 marks)

(5) Let V be a finite dimensional vector space over $\mathbb{C}$. Show that the signature of a Hermitian form $\langle , \rangle : V \times V \to \mathbb{C}$ is independent of the orthogonal basis chosen to write its matrix.
(15 marks)

(6) Let $n > 1$ be an integer. Let $\zeta_n = e^{\frac{2\pi i}{n}}$ and let A denote the $n \times n$ matrix whose entries are $a_{ij} = \frac{\zeta_n^{ij}}{\sqrt{n}}$. Is A a unitary matrix?
(15 marks)

(7) Let $n > 1$ be an integer and A be a real $n \times n$ matrix, where all entries are zero except those on the diagonal and those in the first row and first column. Also assume that all diagonal entries of A are nonzero and that all entries in the first row and first column of A are strictly positive. Show that all the eigenvalues of A are real.
(10 marks) Normal hai!

$\xi_4 = i$

$-1+1+1$
$+-1$
$$\begin{bmatrix} i & -1 & -i & 1 \\ -1 & 1 & -1 & 1 \\ -i & -1 & i & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

(8) Let V be an infinite dimensional vector space over a field F. Let $T : V \to V$ be a linear operator such that $T(V)$, the image of T, is finite dimensional.

(i) Show that T satisfies a nonzero polynomial over F.

(ii) Assume, in addition, that $T^2(V) = T(V)$. Show that $V = \ker(T) \oplus T(V)$.

(15 marks)

(9) This is an **EXTRA CREDIT** question. The solution to this question may depend on *significantly more difficult or different concepts* than what you may have seen during this course. So it is NOT recommended that you attempt and spend time on this question before attempting other questions.

Let V be an infinite dimensional vector space over $\mathbb{C}$ with basis $\mathcal{B}$.

(i) Show that $\mathcal{B}^* := \{v^* : v \in \mathcal{B}\}$ does not span V^*.

(ii) Show that V^* is isomorphic to the direct product of copies of $\mathbb{C}$ indexed by $\mathcal{B}$. Can you say something nontrivial about the dimension of V^* from this?

(20 marks)

$x \in \ker T$

$\langle x, Ty \rangle$

$$\begin{array}{c|c} a & v_1\ v_2 \cdots v_n \\ \hline c_1 & B \\ c_2 & S_{n-1} \\ \vdots & \\ c_n & \end{array}$$

$$\left[\begin{array}{c|c} A & B \\ C & D \end{array}\right]$$

$AD - BC$