HOMEWORK 1 - ALGEBRA I - AUGUST-NOVEMBER 2024

All matrices and objects that appear in this homework are defined over the real numbers.

(1) Define E_{ij} to be the matrix with 1 in row i and column j and 0 in every other entry. Write the product $E_{ij}E_{kl}$.

(2) Find the elementary matrices E_1, E_2, and E_3, whose multiplication on the left of an $r \times c$ matrix M has the same effect as performing the row operations (1), (2), (3) respectively on M. Show that these matrices are invertible.

(3) Show that the number of pivots in any row echelon form of a given matrix M is the same.

(4) Show that reduced row echelon form of a matrix M is uniquely determined.

(5) For what values of c does the system of equations

$$\begin{bmatrix} 1 & 2 & 2 \\ 2 & 4 & 6 \\ 1 & 2 & 3 \end{bmatrix} x = \begin{bmatrix} 1 \\ 4 \\ c \end{bmatrix}$$

will be consistent? Write the general solution for such a c.

(6) Find the intersection in $\mathbb{R}^3$ of the following planes: $x + 2y + 3z = 1$, $2x + 3y + 5z = 2$, $2x + y + 3z = 2$. Plot this intersection.

(7) Find the reduced row echelon form for the following matrix:

$$\begin{bmatrix} -1 & 1 & -1 & -2 \\ 2 & 0 & 0 & 3 \\ -1 & 0 & -1 & -2 \\ -3 & 0 & 2 & -1 \end{bmatrix}.$$

(8) Find an inverse for the matrix

$$\begin{bmatrix} 1 & 2 & 1 \\ 3 & 7 & 3 \\ 2 & 3 & 4 \end{bmatrix}$$

using row operations.

(9) Do the polynomials $x^3 + 2x$, $x^2 + x + 1$, $x^3 + 5$, and $x^3 + 3x - 5$ span $\mathbb{P}_3(\mathbb{R})$? (Here we define $\mathbb{P}_3(\mathbb{R})$ to be the vector space of polynomials in the variable x with degree less than or equal to 3.)

(10) Compute the number of pivots in any row echelon form for the matrix

$$\begin{bmatrix} 1 & 2 & 3 & 1 & 1 \\ 1 & 4 & 0 & 1 & 2 \\ 0 & 2 & -3 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

As a result, write the dimensions of its nullspace and range respectively. Can you find any bases for the nullspace and the range?

HOMEWORK 2 - ALGEBRA I - AUGUST-NOVEMBER 2024

All matrices and objects that appear in this homework are defined over the real numbers.

(1) Give an example of a 6×7 matrix A with dimension of nullspace of $A = 3$, or give an argument why such a matrix cannot exist.

(2) Give an example of a 5×8 matrix A with dimension of nullspace of $A = 2$, or give an argument why such a matrix cannot exist.

(3) Find a basis for the subspace of all vectors in $\mathbb{R}^5$ satisfying

$$x_1 + 3x_2 + 2x_3 - x_4 - 7x_5 = 0.$$

(4) Find a basis for the image and the kernel of the following matrix:

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

(5) For any matrix A, let A^T denote its transpose matrix. Verify the following claim: For A any 2×3 matrix, a vector is in the kernel of A^T if and only if it is perpendicular (in the geometric sense) to the image of A.

(6) Recall that the *trace* of a square matrix is defined to be the sum of its diagonal entries.

- Show that trace is a linear map from the vector space of square matrices to the vector space $\mathbb{R}^1$.
- Show that $\text{trace}(AB) = \text{trace}(BA)$.
- Show that if B is invertible, then $\text{trace}(A) = \text{trace}(BAB^{-1})$.
- Show that the equation

$$AB - BA = \begin{bmatrix} a_0 & & & & \\ & a_1\pi & & & \\ & & a_2\pi^2 & & \\ & & & \ddots & \\ & & & & a_{n-1}\pi^{n-1} \end{bmatrix}$$

  where π is the well-known transcendental ratio of the circumeference of a circle with its diameter and $a_0, \ldots, a_{n-1}$ are arbitrary rational numbers, has no solution in $n \times n$ matrices A, B for any n.

(7) Let X be a finite set. Consider the space $\mathcal{F}(X, \mathbb{R})$ to be the set of functions $f : X \to \mathbb{R}$. Show that this is a vector space under pointwise addition and scalar multiplication. What is the dimension of this space if X is finite?

(8) Classify all the linear transformations T in the following cases:

- $T : \mathbb{R} \to \mathbb{R}$.
- $T : \mathbb{R} \to \mathbb{R}^2$.
- $T : \mathbb{R}^2 \to \mathbb{R}$.
- $T : \mathbb{R}^2 \to \mathbb{R}^2$.

What does this say geometrically?

HOMEWORK 3 - ALGEBRA I - AUGUST-NOVEMBER 2024

(1) A matrix B is *symmetric* if $B = B^t$, where ^{t} denotes the transpose of the matrix.

(i) Show that for any square matrix B, $B + B^t$ and BB^t are symmetric.

(ii) If A is invertible, then $(A^{-1})^t = (A^t)^{-1}$.

(iii) Let A and B be symmetric $n \times n$ matrices. Prove that the product AB is symmetric if and only if $AB = BA$.

(2) This exercise is about determinants of block matrices.

(i) Show that the determinant of the matrix

$$M = \begin{bmatrix} A & B \\ 0 & D \end{bmatrix}$$

is computed as $\det M = (\det A)(\det D)$, if A and D are square blocks.

(ii) Let a $2n \times 2n$ matrix M be given in the form

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

Suppose that A is invertible and that $AC = CA$. Use block multiplication to prove that $\det M = \det(AD - CB)$. Give an example to show that this formula need not hold if $AC \neq CA$. Does the formula hold if A is not invertible but $AC = CA$?

(3) Determine the smallest integer n such that every invertible 2×2 matrix can be written as a product of at most n elementary matrices.

(4) Let $x_1, x_2, \ldots, x_n$ be variables. Compute the determinant of the matrix

$$\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ x_1 & x_2 & x_3 & \cdots & x_n \\ x_1^2 & x_2^2 & x_3^2 & \cdots & x_n^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_1^{n-1} & x_2^{n-1} & x_3^{n-1} & \cdots & x_n^{n-1} \end{bmatrix}.$$

(5) Let V denote the space $\mathbb{R}_{n\times n}$ of all $n \times n$ real matrices.

(i) V is the direct sum of the space of symmetric matrices and the space of *skew-symmetric* matrices, where skew-symmetric matrices are defined as the matrices A satisfying $A^t = -A$.

(ii) Let W denote the subspace of V of matrices with trace zero. Find $W' \subset V$ a subspace such that $V = W \oplus W'$.

(6) Let V be a vector space over an infinite field F. Prove that V is not the union of finitely many proper subspaces.

HOMEWORK 4 - ALGEBRA I - AUGUST-NOVEMBER 2024

(1) Recall that a linear operator T is *nilpotent* if some positive power T^k is zero. Prove that T is nilpotent if and only if there is a basis of V such that the matrix of T is upper triangular, with diagonal entries zero.

(2) Let

$$M = \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}$$

be a matrix in block form. Show that M is diagonalizable if and only if A and D are diagonalizable.

(3) Let M be a 2×2 matrix with eigenvalue λ.

(i) Show that unless it is zero, the vector $(b, \lambda - a)^t$ is an eigenvector.
(ii) Find a matrix P such that $P^{-1}MP$ is diagonal, assuming that $b \neq 0$ and that M has distinct eigenvalues.

(4) Let M be the $n \times n$ matrix given as

$$\begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

(i) Show that $M^n = 0$ but $M^{n-1} \neq 0$.
(ii) Find all $n \times n$ matrices N over complex numbers that commute with M, i.e. $MN = NM$.

(5) Consider the complex numbers $\mathbb{C}$ as a 2-dimensional vector space V over the real numbers $\mathbb{R}$. For any complex number α, consider the function $T_\alpha : V \to V$ given by $v \to \alpha v$ obtained by simply multiplying by α. Using the basis $1, \iota$ of complex numbers over real numbers, find matrix of T_α. Show that the set $\{T_\alpha\}$ satisfies all properties of complex numbers that you know.

(6) If A and B are two square matrices which are both diagonalizable such that $AB = BA$, then show that there exists an invertible matrix C such that CAC^{-1} and CBC^{-1} are both diagonal.

(7) Let V be a vector space over a field F and let W be a subspace. Define

$$A(W) := \{f \in \hat{V} : f(w) = 0 \text{ for all } w \in W\}.$$

(i) Show that $A(W)$ is a subspace of $\hat{V}$ and that if $W \subset U$, then $A(U) \subset A(W)$.
(ii) Show that dimension of $A(W)$ equals $\dim V - \dim W$.
(iii) Show that $A(A(W)) = W$.

(8) Let V be an inner product space over $\mathbb{R}$ or $\mathbb{C}$. Define distance $d(u, v) := \|u - v\|$. Show that it satisfies the triangle inequality.

(9) If $\{w_1, w_2, \ldots, w_n\}$ is an orthornomal set in an inner product space V, show that

$$\sum_i |\langle w_i, v \rangle|^2 \leq \|v\|^2$$

for any $v \in V$.

HOMEWORK 5 - ALGEBRA I - AUGUST-NOVEMBER 2024

(1) In this question, the matrices are defined over complex numbers. $n > 1$ is an integer.

(i) Show that a polynomial of degree n defined over any field F has at most n roots.
(ii) Show that the set of $n \times n$ matrices commuting with a fixed $n \times n$ matrix A is a vector subspace of $\mathbb{C}^n$. Compute the dimension of this space.
(iii) Show that if A is an $n \times n$ matrix which has n distinct eigenvalues, and B is an $n \times n$ matrix such that A and B commute, then B is a polynomial of degree at most $n-1$ in the matrix A. That is, there exists a polynomial $p(x)$ with complex coefficients and degree at most $n-1$ such that $p(A) = B$.

(2) Show that a bilinear form $\langle , \rangle$ on a real vector space V is a sum of a symmetric form and a skew-symmetric form.

(3) Let A and A' be real symmetric matrices such that $A' = P^t AP$, where P is an invertible matrix. Are ranks of A and A' equal?

(4) Show that the set of $n \times n$ Hermitian matrices forms a real vector space and compute its dimension.

(5) Show the following if true, or demonstrate that it is false by finding a counterexample and salvage the result if possible by adding a suitable hypothesis or weakening the conclusion :

(i) If A is a real $m \times n$ matrix, $B = A^t A$ is a symmetric positive definite matrix of the same rank as A.
(ii) If A is a complex $m \times n$ matrix, $B = A^* A$ is a unitary positive definite matrix of the same rank as A.

(6) Show that the signature of a bilinear form $\langle , \rangle$ is independent of the orthogonal basis chosen to write its matrix.

(7) Let $V = \mathbb{R}_{2\times 2}$ be the real vector space of real 2×2 matrices. Let $W = \mathbb{C}_{2\times 2}$ be the real vector space of complex 2×2 matrices.

(i) Show that the function $\langle , \rangle : V \times V \to \mathbb{R}$ given by $\langle A, B\rangle = \text{Trace}(AB)$ is a symmetric bilinear form. Determine its matrix with respect to the standard basis. Determine its signature.
(ii) Show that the function $\langle , \rangle : V \times V \to \mathbb{R}$ given by $\langle A, B\rangle = \text{Trace}(A^t B)$ is a symmetric bilinear form. Determine its matrix with respect to the standard basis. Determine its signature.
(iii) Is the function $\langle , \rangle : W \times W \to \mathbb{C}$ given by $\langle A, B\rangle = \text{Trace}(A^* B)$ a Hermitian form? If so, determine its signature.
(iv) Is the function $\langle , \rangle : W \times W \to \mathbb{C}$ given by $\langle A, B\rangle = \text{Trace}(\bar{A} B)$ a Hermitian form? (The $\bar{A}$ denotes the complex conjugation applied to all entries of the matrix A.) If so, determine its signature.

(8) Let V be the real vector space of 3×3 matrices equipped with the bilinear form $\langle A, B\rangle = \text{Trace}(A^t B)$ and let W denote the subspace of skew-symmetric matrices. Compute the orthogonal projection to W of the matrix

$$M = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 1 & 3 & 0 \end{bmatrix}$$

with respect to this form.

(9) For a Euclidean space V, prove the parallelogram law:

$$\|v + w\|^2 + \|v - w\}^2 = 2\|v\|^2 + 2\|w\|^2.$$

Does it hold in a Hermitian space?

(10) Let $w \in \mathbb{R}^n$ be a vector of length 1 under the usual dot product. Recall that for any vector v, we can uniquely write $v = cw + u$ where $u \in W^\perp$ and c is a real number. Define $r_w(v) := -cw + u$.

(i) Prove that the matrix $P = I - 2ww^t$ is orthogonal, i.e. $PP^t = I$.
(ii) Prove that multiplication by P is a reflection about the space $W^\perp$. Is this the same map as r_w?
(iii) Show that the linear operator given by P preserves lengths of vectors.
(iv) If v, v' are two vectors of equal length, find a vector w such that $Pv = v'$.

(11) Let $T : V \to V$ be a linear transformation on $V = \mathbb{R}^n$ such that its matrix in some basis is a real symmetric matrix. Prove that $V = \ker(T) \oplus \operatorname{im}(T)$.

(12) Let T be a unitary operator on a Hermitian space $(V, \langle , \rangle)$ and let v_1, v_2 be two eigenvectors of T with distinct eigenvalues. Show that v_1 and v_2 are orthogonal.