MIDTERM EXAMINATION - ALGEBRA I - AUGUST-NOVEMBER 2024

Instructions: The total time allotted for this examination is 3 hours. This is a closed book examination. Books, notebooks, cellphones, laptops and any such objects that may enable you to get external help are not allowed. Be brief but precise in your answers. You should justify your assertions. The maximum marks that can be scored on this exam are 100.

All matrices that appear in this question paper are real matrices unless otherwise specified.

(1) Let a, b be real numbers. Consider the following system of equations:

$$X + Y + 2Z = a,$$
$$2X + 2Y + 3Z = b,$$
$$3X + 3Y + 4Z = a + b.$$

(i) Determine all possible values of a, b for which the above system has a solution. When the system has a solution, describe all solutions in terms of a and b.

(ii) Are there any real numbers a, b for which the system of equations above has exactly one solution?

(10 marks)

(2) Compute the ranks of the following matrices:

(i)

$$\begin{bmatrix} 1 & 2 & 3 & \cdots & 10 \\ 11 & 12 & 13 & \cdots & 20 \\ 21 & 22 & 23 & \cdots & 30 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 91 & 92 & 93 & \cdots & 100 \end{bmatrix}.$$

(ii)

$$\begin{bmatrix} 1 & 2 & 3 & \cdots & 10 \\ 2 & 4 & 6 & \cdots & 20 \\ 3 & 6 & 9 & \cdots & 30 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 10 & 20 & 30 & \cdots & 100 \end{bmatrix}.$$

(15 marks)

(3) Let $P_{\leq n}$ be the vector space of polynomials $p(x)$ in single variable x of degree less than or equal to n. Show that taking the derivative defines a linear map $D : P_{\leq n} \to P_{\leq n-1}$ and write down its matrix in two bases: $\mathcal{B} = \{1, x, x^2\}$ and $\mathcal{B}' = \{1, (x+1), (x^2 + x + 1)\}$.

(5 marks)

(4) Let n be a natural number greater than 1 and let M denote any $n \times n$ square matrix. Show that for any invertible $n \times n$ matrix P, trace of M = trace of PMP^{-1}.

(5 marks)

(5) Let n be a natural number greater than 1. Define an $n \times n$ matrix M to be *nilpotent* if there exists some $k \in \mathbb{N}$ such that $M^k = 0$, the zero matrix (i.e. the $n \times n$ matrix whose entries are all 0).

(i) What can you say about kernels of the matrices $M, M^2, M^3, \ldots$ when M is a nilpotent matrix?

(ii) Show that trace of a nilpotent matrix M is 0.

(iii) Give an example of a trace 0 matrix (for every n) that is not nilpotent.

(15 marks)

(6) Give an example of a 3×3 matrix M such that trace of M = trace of $M^2 = 0$, but trace of M^3 is not 0.

(15 marks)

(7) Let n be a natural number greater than 1. Let M be an $n \times n$ matrix whose trace is 0. Let R_θ denote the matrix of rotation by an angle θ in the euclidean plane $\mathbb{R}^2$.

(i) For a 2×2 matrix

$$A = \begin{bmatrix} a & u \\ v & b \end{bmatrix},$$

compute the effect of conjugating this matrix by R_θ.

(ii) If a and b have opposite signs, show that there exists a matrix Q such that QAQ^{-1} has upper left entry 0.

(iii) Show that there exists an invertible $n \times n$ matrix P such that PMP^{-1} has all 0s on the diagonal.

(iv) Show that an $n \times n$ matrix N that has all 0s on the diagonal can be written as $N = XY - YX$ for some $n \times n$ matrices X and Y.

(v) Show that M can be written as $M = AB - BA$ for some $n \times n$ matrices A and B.

(25 marks)

(8) Let n be a natural number greater than 1. Let T be an $n \times n$ matrix such that $T^2 = T$. T is called *idempotent* in this case.

(i) Prove or disprove: $1 - T$ is also an idempotent.

(ii) Show that the column spaces of T and $1 - T$ are disjoint from each other, and that they span $\mathbb{R}^n$.

(10 marks)

$T(x) = a_1 v_1 + a_2 v_2 + \dots + a_k v_k$

$$\sum_{j=1}^{3} \sum_{i=1}^{3} a_{ij} a_{ji} = \sum_{i=1}^{3} a_{ii}$$

$a_{11} + a_{22} + a_{33} = 0$

$$\begin{bmatrix} 1 & 1 & -2 \\ 1 & 1 & -2 \\ 1 & 1 & -2 \end{bmatrix}$$