Class Notes

Class: Algebra I

Professor: Aditya Karnataki

ONE OF THE MOST IMPORTANT COURSES! WILL COME UP IN EVERY PLACE: Math, Engineering, Economics, ... maybe even poetry.
A simple way to be good at linear algebra is to think straight!
Throughout the course the emphasis will be on linear algebra over real and complex fields.
Topics covered will be - Systems of linear equations, row reduction, vector spaces over any field, bases and dimension, change of basis, linear transformations, dimension formula, matrix of a linear transformation and change of bases, linear operators, similarity, determinants and invertibility, eigenvalues and eigenspaces, characteristic polynomial, triangular and diagonal forms, Euclidean/Hermitian spaces with standard inner product, orthogonal projection, orthonormal bases and Gram-Schmidt procedure, spectral theorem for Hermitian/symmetric operators.
The above material corresponds to the following from Artins Algebra: chapters $1, 3, 4.1 - 4.6$, parts of chapter 8.
Grades will be based on a final exam (40%), a midterm exam (30%), and in-class quizzes and assignments (30%). Homework will be given but not submitted unless specified. Quizzes and exams will likely include homework-related questions, so it's important to complete assignments regularly. Quiz dates will generally be unannounced, and there are no make-up quizzes or late homework submissions. However, Aditya sir will drop the lowest 1-2 quiz scores for everyone, depending on the total number of quizzes.

July 18, 2025

Contents

1 Aditya Karnataki sir's Knowledge

To be set in CMI, be good at Linear Algebra, Calculus, Statistics and Programming.

I can teach you how to bewitch the mind and ensnare the senses. I can tell you how to bottle fame, brew glory, and even put a stopper in death. Snape meant it about potions, but I mean it about linear algebra.

Solve all problems. Solving problems is math, talking about math is not math!

You are all here as you like doing mathematics. No one begins math by thinking about a career in it, you do it as it gives you joy. You will remember your exams and the problems which you solved ingeniously and it will give you joy. Cling to the joy, and don't let it go!
Because this is what math is.

We could have mathematica or desmos do it, but we wish to do our own math. Not let others do it for us.

Tommorow is independence day, but we will still have a class, as there is no independence form mathematics. And why would we even want that?

You all always steal my punchline. All of you would be very bad stand-up audience.

There are no questions which are silly. Except "Who am I?" which philosophers have anyhow been asking for centuries. They give silly answers as well. Like "I am he". Barbie had a better answer then them, "I am a Barbie girl!"

Pivots play a pivotal role.

There is a wide wide world to explore. And all of it will need linear algebra!

Some questions seem like existential questions but sometimes we don't need to answer them. We don't need to wait for an angel to descend from heaven to answer it, we can just say, for me the answer is this and it works, as long as it makes sense.

Good attendance. That means I am a good teacher, atleast that is what I tell myself when I lay in bed at night, in despair.

Do not get caught in the quotes of great mathematicians. Everyone has their own journey.

[written on board] Nudge nudge wink wink

The one basis to span them all. <Dead silence> Lord of the Rings. Like I am the lord of the jokes. That way someone can throw something into a volcano and all my jokes will go away.

I wrote one of my papers by choosing a different basis. So even that came down to linear algebra. Sometimes math comes down to such choices. SO you should be willing to make such choices. What if people worked with some basis for 40 years? This one makes my work easier. Although, it sometimes makes your work harder as well, people used the basis for 40 years for a reason.

Yes, we can! that's what Obama said in 2008. Wait, you all too young for that.

It is okay to be stupid. Oh, I meant it is okay to feel stupid.

I can't see. I have washed my glasses with ragda pattis.

As Sherlock says to Watson, "It is simple." <some guy from class "Trivial">, We don't use that word here. If I see it anywhere in your quiz or exam answer, it is 0, trivially.

<some guys alarm rings> Come on! You are allowed to sleep in my class but putting an alarm? That's just insulting. I will tell you when I am going, I am a better alarm.

If I want to see if I can lift weights, I go and lift ten kilos. I don't life hundred kilos. That won't happen, atleast not overnight.

I suggest you keep philosphical discussions about "Axiom of Choice" away till you becoe a proffesor or get a job. That is unless you want to enter set theory and formal logic. In that case, your getting a job may depend on you thinking about this. However, for my class, we belive in axiom of choice. See my job is to do math, not talk about it. Like I like talking about it, that is why I am taking this class. But if taking axiom of choice allows me to work with infinite vector spaces, which is more rewarding, I will accept that.

Mein lagta hoon utna bewkoof nahi hoon (I am not as foolish as I look).

It is said that in CMI, you can easily miss the boat. However, if you are already on the boat, how can you miss it? Just like how math is locally trivial, so is study and preperation.

Linearity should be by now, fami-linear?

<what does your t-shirt say?>
It is just a dragon ball t-shirt. Any of you know dragon ball? It is the tshirt Goku wears. It reads 'Go' which means wisdom. As I don't have it in my head, I have it on my T-shirt. I hope none of you read it as 'Go' and understood that attendence is over and now want to go.

<How is the pattern of the exam?>
I don't know. I am sure it must be doing well.

About the exam pattern, I am as clueless as you are. Sometimes even more.

ψ and ϕ. Psi and phi. There will be a lot of psi and a lot of phi. A lot of sight and a lot of fight, hence it is sci-fi. I hope you have the sight to see and it and the fight to fight it.
This joke is quite light.
This is what I think about at night.
Let's return to the blackboard, which is not white.
Right?

Let there be no trace of midterms in your mind.

Let that not be a determinent of what happens in this class.

This class and the midterm are linearly independent.

<sir, no fans!> I see, I don't have fans in this class. But I am not here to collect fans, I am here to do linear algebra.

There may be an analysis questions in algebra exam. There may be a number theory question in your topology course. It's not a me issue, it is a math issue. Math doesn't say this question has only linear algebra.

I have issues with number theory. Why else do you think I am balding so much? I pull my own hair.

Find the isomorphism of sections from Artin to Hoffman-Kunze. I will depend on these coordinates.

Sometimes math is unmotivated. The motivation arrives late, but till then your perseve and subsist and find a thesures and find synonyms of perseve and subsist.

It all depends on what you remember and what you forget. I am a very forgetful person. THere will also be forgetful functors , when you go to category theory. So there are other things than forgetful teachers,

We have exited the realm of real numbers and entered the complexity of complex numbers. Don't complexify the situation. I hope I didn't give you all a complex.

Algebric multiplicity. AM.
Geometric multiplicty. GM.
So $AM \geq GM$. (laughs manically like an evil scientist who just made a 'breakthrough')

Not everything can be taught in the class, this is what you must learn from this class.

Questions in real life will not ask you if you learnt this in the course on linear algebra. You just need to learn it.

Symbols appear on the white board when you write them with white chalk... or colored chalk. But we must know what these symbols mean.

<walks in while discussing Julias Caeser> So many people today. Play reharsal? Ah, exactly 23 people. Please keep weapons outside. Who is Brutus anyway?

<Aditya Vashishth interupts> Hey, only one Aditya may speak at a time.

Pi is overloaded. Actually, P is overloaded. Actually, all letters are overloaded.
You only get small pi.

The chalk didn't chalk.
Chalk the Chalk on the Chalk(chowk?).
You have to chalk the talk. Like Walk the talk.
Like, you have heard Walk the Talk, right?
Like you have to walk
You can't just talk.

<shouting> REMEMBER, REMEMBER. DO NOT FORGET!
DO NOT FORGET THE FIFTH OF NOVEMBER!
<What?>
That reference is lost to you all. It is from 'V For Vendetta'

Hermitian forms are not called Hemitian as they are hermits in the world of forms. They are not socially hermits. They are actually 'Hormit'.

The choice of bar on variable is purely a choice. It has no Inner, pun intended, effects.

DON'T PANIC!
As is said on the first page of Hitchiker's guide said.
I'll probably put that on the first page of your exam sheet.
<That would probably feel like a mock>
So I'll write "PANIC"
<Um...>
There's no pleasing you all. I'll just write FEEL FREE TO PANIC OR NOT.
Wait, that's a tautology.

Sometimes you come to an answer, then feel it can't be write and write a completely brand new, wrong answer. Don't do that. Trust your intuition.
<Doesn't our intuition only cause us to do so?>
Um, so have you guys seen Inside Out 2? Don't listen to the orange guy.
Wait, I am wearing orange.

<Everything is basic.> Yah, everything is basic, only I am acidic.

A real vector space is on reals.
It is real man!
So real that it is complex.

Spectral Theorem.
This theorem will haunt you for rest of your life.
Specter?
It will give you nightmares.
So hold your horses.
Horses, mares?

The TAs have informed me that my bad humor has rubbed off you all. Although, I don't pay much heed to them. They are rather humorless.

Midway switching notation.
Good for health.
Neither yours, Neither mine.
Thankfully, I don't get marks.

2 Why study linear algebra?

Linear algebra is the study of linearity. It may be hidden, is very well hidden in most cases, but once found, we can pounce with linear algebra. Linear equations are the simplest. I can't drive a F1 car, but I can use my ideas of driving cycle to understand F1 racing. Even when the situation is not linear, we can use linearity to understand. Similar to how a tangent line of a curve can approximate a function.

Definition 2.1. *! in a math statement means 'unique'*

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Definition 3.1. *For now,*

$$\mathbb{R}^n = \{[v_1, v_2, \ldots, v_n]^T \mid v \in \mathbb{R}\}$$

is the set of vectors in n-dimensional space.
An n-coordinate vector is an element of $\mathbb{R}$*.*
This is denoted as $\vec{v}$ *and* $\vec{w}$ *but later only as* $\mathbf{v}$ *and* $\mathbf{w}$*.*

3.1 Addition on vectors

Addition is performed entrywise for both row and column vectors.

$$+ : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$$

The obvious properties of addition (using geometry and/or definition) are:

$$\vec{v} + \vec{w} = \vec{w} + \vec{v}$$

$$(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$$

$$\exists \vec{0} \text{ such that } \vec{v} + \vec{0} = \vec{v} \implies \vec{0} = [0, 0, \ldots, 0]$$

The negative of $\vec{v}$ exists and satisfies $\vec{v} + (-\vec{v}) = 0$, which implies that $-\vec{v} = [-v_1, -v_2, \ldots, -v_n]$.

3.2 Scalar multiplication

Scalar multiplication is performed entry-wise for both row and column vectors.

Definition 3.2.

$$\cdot : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$$

The obvious properties are:

$$1 \cdot \vec{v} = \vec{v}$$

$$(c_1 c_2)\vec{v} = c_1(c_2\vec{v})$$

$$(c_1 + c_2)\vec{v} = c_1\vec{v} + c_2\vec{v}$$

$$c(\vec{v} + \vec{w}) = c\vec{v} + c\vec{w}$$

3.3 We change the definition!

Definition 3.3. *A vector space over real numbers is a set V with two operations such that*

$$+ : V \times V \to V$$
$$\cdot : \mathbb{R} \times V \to V$$

satisfying all the properties listed above.

With this, we finally introduce the definition of a field.

Definition 3.4. *Informally, a set of "numbers" where you can add, subtract, multiply, and divide by anything non-zero is a field.*

Doing so formally,

Definition 3.5 (Artin 3.2)**.** *A field is a set $\mathbb{F}$ together with $(+, 0, -, \cdot, 1, ()^{-1})$ where:*
- $0, 1 \in \mathbb{F}$
- $+, \cdot$ *are functions* $\mathbb{F} \times \mathbb{F} \to \mathbb{F}$
- $-$ *is a function* $\mathbb{F} \to \mathbb{F}$
- $()^{-1}$ *is a function* $\mathbb{F} - \{0\} \to \mathbb{F} - \{0\}$

satisfying:
- $a + b = b + a$
- $(a + b) + c = a + (b + c), \quad (a \cdot b) \cdot c = a \cdot (b \cdot c)$
- $a + 0 = a, \quad a \cdot 1 = a$
- $a + (-a) = 0$
- $a \cdot a^{-1} = 1$
- $a(b + c) = ab + ac$

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4.1 Recall

Vectors were defined as:

$$\mathbb{R}^n = \left\{ \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \mid v_i \in \mathbb{R} \right\}$$

where the elements of $\mathbb{R}^n$ are called n-dimensional vectors. Here, $\mathbb{R}^n$ is the n-dimensional vector space.

A vector space is defined by the following properties:
- Any vectors of the "same size" can be added or subtracted.
- You can scale any vector by a real number.

4.2 Matrices

Matrices are arrays of numbers[1].

An $r \times c$ matrix has r rows and c columns. We can multiply an $r \times c$ matrix with a column vector in $\mathbb{R}^c$[2].

More generally, we can multiply an $A_{p\times q}$ matrix with a $B_{q\times r}$ matrix to get the product AB, which is a $p \times r$ matrix[3]. Note that BA may not exist and only exists if $p = r$.

We can add matrices of the same size element-wise and also scale matrices (multiply by a scalar) element-wise.

Notice that these are the same properties as vectors, except that we can also multiply matrices.

[1] Actually, they are linear transformations, not just arrays of numbers.
[2] This is a linear transformation from $\mathbb{R}^c$ to $\mathbb{R}^r$.
[3] Why isn't the instructor focusing on linear transformations?

4.3 Properties of Matrix Multiplication

Why don't we multiply vectors element-wise? The simple answer is that it is a useless operation:

$$\begin{bmatrix}1\\0\end{bmatrix} \odot \begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

This shows that multiplying a vector on the x-axis with another on the y-axis gives the zero vector, which doesn't provide meaningful information. Rotating this by 45° gives us:

$$\begin{bmatrix}-1\\1\end{bmatrix} \odot \begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}-1\\1\end{bmatrix}$$

This is a senseless operation to begin with. Hence, we don't define vector multiplication this way[4].

Now, let's consider the properties of matrix multiplication as it is normally defined:

- $(AB)C = A(BC)$
- $(A+B)C = AC+BC$
- $C(A+B) = CA+CB$
- The identity matrix $I_{p\times p} = I_p$ is a square matrix with 1s on the diagonal ($i = j$) and 0s elsewhere.

4.4 Systems of Linear Equations

We can use matrices to define a system of linear equations. For example:

$$6x - y = 35x + 2y = 10$$

can be written as:

$$\begin{bmatrix}6 & -1\\5 & 2\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}3\\10\end{bmatrix}$$

This means we can use matrices to solve equations. Let's first talk about simple cases.

4.4.1 Small Cases: 1 Equation in 1 Variable

$$3x = 6 \qquad \text{(1 solution)}$$
$$0x = 6 \qquad \text{(no solution)}$$
$$0x = 0 \qquad \text{infinitely many solutions}$$

We can now expand this to more variables:

# of Variables	# of Equations	Picture	# of Solutions
2	1	Line	∞ (1-dimensional)
	2	Intersection of two lines	
		(1) Same line	∞ (1-dimensional)
		(2) Parallel lines	0
		(3) Intersect at one point	1
3	1	Plane	∞ (2-dimensional)
	2	Intersection of two planes	
		(1) Same plane	∞ (2-dimensional)
		(2) Parallel planes	0
		(3) Line	∞ (1-dimensional)
	3	Intersection of 3 planes (P, Q, R)	
		$l \in R$	∞ (1-dimensional)
		$l \| R$	0 (0-dimensional)
		$l \cap R$	1 (0-dimensional)

[4] We define multiplication in terms of linear transformations. Just say it!

4.5 How Do We Solve Equations?

- Reorder equations.
- Scale an equation by a non-zero scalar.
- Add a multiple of one equation to another.

All of this can be done using the matrix representation of $Ax = B$ or the augmented matrix $[A \mid B]$, which can be row-reduced.

4.6 Solving an Example

Given the equation:

$$2x + 4y + 6z + 8w = 1$$

we can solve for x:

$$x = \frac{1}{2}(1 - 4y - 6z - 8w)$$

This gives us the solution set:

$$\left\{ x = \frac{1}{2}(1 - 4y - 6z - 8w) \mid y, z, w \in \mathbb{R} \right\}$$

Here, y, z, and w are called free variables, while x is the pivot variable.

Now consider:

$$\begin{aligned} 2x + 4y + 6z + 8w &= 1, \\ 3x + 7z + w &= 10 \end{aligned}$$

This corresponds to the matrix:

$$\begin{bmatrix} 2 & 4 & 6 & 8 & \mid & 1 \\ 3 & 0 & 7 & 1 & \mid & 10 \end{bmatrix}$$

We can eliminate x from the second row by performing the operation $R_2 - \frac{3}{2}R_1$ to get:

$$\begin{bmatrix} 1 & 2 & 3 & 4 & \mid & \frac{1}{2} \\ 0 & -6 & -2 & -11 & \mid & \frac{17}{2} \end{bmatrix}$$

Remark 4.1. *We can solve as follows:*

1. *Take w, z arbitrarily.*
2. *Solve for y using these values in the second equation.*
3. *Solve for x using the values of y, z, w in the first equation.*

Remark 4.2. *We can further simplify by reducing the matrix to remove y:*

$$\begin{bmatrix} 1 & 0 & \frac{7}{3} & \frac{1}{3} & \mid & \frac{40}{12} \\ 0 & 1 & \frac{1}{3} & \frac{11}{6} & \mid & \frac{-17}{12} \end{bmatrix}$$

As we have the largest possible identity submatrix, we call this the row-echelon form.

Note that the solution space has dimension 2.

Definition 4.3 (Preliminary)**.** *The solution space has a dimension equal to the number of free variables.*

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5.1 Recall

We performed row reduction on matrices last time. The most row-reduced matrix, with a sub-matrix being an identity matrix, is called the row-echelon form.

5.2 General Procedure

1. Reorder the equations so that the leftmost entry appears in the first row of the augmented matrix.
2. Scale the first equation of the first row to make this entry equal to 1.
3. Make all entries below this entry 0 by subtracting appropriate multiples of the first row.
4. Continue this process from row 2 onward.

5.3 Shape of the Result

$$\begin{bmatrix} 0 & 0 & \dots & 0 & | & 1 & * & * & \dots & * & | & * & * & \dots & * \\ 0 & 0 & \dots & 0 & | & 0 & 0 & 0 & \dots & 0 & | & 1 & * & \dots & * \\ 0 & 0 & \dots & 0 & | & 0 & 0 & 0 & \dots & 0 & | & 0 & * & \dots & * \\ \vdots & \vdots & \vdots & \vdots & | & \vdots & \vdots & \vdots & \vdots & \vdots & | & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & | & 0 & 0 & 0 & \dots & 0 & | & 0 & * & \dots & * \end{bmatrix}$$

The leading non-zero entry in any row is called a pivot.

Remark 5.1. *"Want to do better"*
We can also make entries above each pivot 0.

Note: The operations used are all reversible. This means that we can undo the change from the row-reduced to the original form.

Example 5.2. *Performing an elementary row operation on a matrix M is the same as multiplying M on the left by a suitable matrix. Find these matrices.*
The operations are switching two rows, scaling a row, and adding one row to another.

Solution. Switching two rows is defined by:

$$\begin{bmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 \end{bmatrix}$$

This particular one switches rows 2 and 3. In the general case, this is simply an identity matrix with the m^{th} and n^{th} rows interchanged.
Scaling a row is defined by:

$$\begin{bmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & n & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 \end{bmatrix}$$

This particular matrix scales row 3 by n. In the general case, this is just the identity matrix with n in place of 1 in the m^{th} row, causing the m^{th} row to be scaled by n.
The addition of rows is defined by:

$$\begin{bmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ 1 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 \end{bmatrix}$$

This particular matrix adds row 4 to row 1. In the general case, this is simply the identity matrix with some $(i, j) = 1$ where $i \neq j$. This will cause the j^{th} row to be added to the i^{th} row. ☐

5.4 Row Echelon Form for M

Definition 5.3. *A matrix M after a series of operations satisfies the following properties:*

- *Pivots in successive rows should move strictly to the right (so all entries below a pivot must be* 0*).*
- *All rows without pivots must come at the bottom.*
- *It is customary to have pivots equal to* 1.

We already saw the general shape of it before in an aptly titled section.

5.5 Reduced Row Echelon Form

Definition 5.4. *A row-echelon matrix with the additional property that all entries above a pivot are* 0.

Example 5.5. *Is the row-echelon form of a given matrix M unique?*

Proof. This is simply untrue. Just add any of the non-pivot rows to each other.
Since not every row-echelon form is reduced, but every reduced form is row-echelon, this in itself disproves the statement. □

Example 5.6. *Prove that the Row Reduced Echelon Form is unique for a given matrix M.*

Proof. We will solve using induction. It is trivial to show that this works for an $m \times 1$ matrix.
Assume this is true for an $m \times (n-1)$ matrix. Then we will prove it for an $m \times n$ matrix.
FTSOC, let a matrix A of size $m \times n$ have two RREFs, B and C such that $B \neq C$.
Then by the inductive hypothesis, the difference may only lie in the last row. Let's consider a system of equations such that $AX = 0 \implies BX = CX = 0$ where X is a variable vector $[x_1, x_2, \dots, x_n]^T$.
This implies that $(B - C)X = 0$. Notice that the first $n-1$ columns will be 0, but as $B \neq C$, we will have at least one element $b_{n,j} - c_{n,j} \neq 0$.
This means we will have $x_n = 0$ using simple vector multiplication.
This implies that the last variable is fixed at 0. This means there must be a pivot in the last column, and since we are in an RREF, the pivot location is deterministic.
Thus, the last column of B and C is the same, giving $B = C$.
However, this is a contradiction, and thus, our initial assumption is false. Therefore, any matrix A of size $m \times n$ has a unique RREF.
By induction, all matrices have a unique RREF. □

Example 5.7. *Prove that the number of pivots in any echelon form is the same for a given matrix M.*

Proof. TBD □

Definition 5.8. *A system of linear equations with at least one solution is called consistent. Having infinitely many solutions is also considered consistent.*

Theorem 5.9. *$A\vec{x} = \vec{b}$ is inconsistent precisely when there is a pivot in the last column of the Row Echelon Form of the augmented matrix $[A \mid \vec{b}]$.*

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6.1 Recall

Last we left off, we had studied that $Ax = B$ is consistent if there is a solution.
$Ax = B$ is consistent for all $B \in \mathbb{R}^r$ if and only if there is no pivot in the last column of $[A \mid B]$.
This also has an if and only if relation with all rows of RREF of A having a pivot.
This also has an if and only if relation with the number of pivots being r.

6.2 Homogeneous Equation

$Ax = 0$ is always consistent as $x = 0$ is always a solution for any A. The non-trivial solutions are interesting in their own way.
Let's prove a fact.

Example 6.1. *Given*

$$S := \{x \mid Ax = b\}$$

Let v be given such that $Av = b$.

$$T := \{v + y \mid Ay = 0\}$$

Prove that $S = T$.

Proof. We will prove this by proving (I) $S \subseteq T$ and (II) $T \subseteq S$, which will imply that $S = T$.
(I) Let x be such that $Ax = b$. This implies $x = v + y \implies y = x - v$.

$$\begin{aligned} Ay &= A(x-v) \\ &= Ax - Av \\ &= b - b \\ &= 0 \end{aligned}$$

Thus, $S \subseteq T$.
(II) Let $u \in T$ such that $u = v + y$, $Ay = 0$. We need to show that $Au = b$. Notice:

$$\begin{aligned} Au &= Av + Ay \\ &= b + 0 \\ &= b \end{aligned}$$

Thus, $T \subseteq S$.
And as $S \subseteq T$ and $T \subseteq S$, we have $S = T$ and we are done! □

We will also now observe that the number of pivots in RREF of a matrix $A_{r\times c}$ has at most $\min(r, c)$ pivots.
This implies that saying "$Ax = 0$ has only the trivial solution for a particular matrix A" is equivalent to saying "Every column of the REF of A has a pivot".
Another strange fact is that $Ax = b$ for any b will either be inconsistent or the solution set will have the same dimension as the solution set of $Ax = 0$.

Theorem 6.2. *For a square matrix A, $Ax = 0$ has a unique solution $\iff$ $Ax = b$ has a solution for every $b \in \mathbb{R}^r$ (which then must be unique).*

Example 6.3. *Prove that two of these imply the third.*
(1) A is a square matrix.
(2) $Ax = 0$ has a unique solution.
(3) $Ax = b$ has a solution for every $b \in \mathbb{R}^r$.

6.3 Matrix Inverse

We take the row operation matrices to be invertible.
We can notice that for most matrices A such that $\text{RREF}(A) = I = E_1^{-1}E_2^{-1}E_3^{-1}\dots A$,
this implies that A is invertible and its inverse is $E_1^{-1}E_2^{-1}E_3^{-1}\dots$.

6.4 Matrices as Linear Transforms (Yay!)

$$f : \mathbb{R}^c \to \mathbb{R}^r$$

This function can also be written as

$$f : x \to Ax$$

where A is a $r \times c$ matrix and x is a column vecotor with c columns. This makes Ax a column vector with r columns.
We should also note that $Ax = b$ has a solution if and only if b is in the range(or image) of f.
The normal set theoretic definations still hold. That is:

Definition 6.4. *$Ax = b$ has a solution for a all b if and only if f is surjective.*

Definition 6.5. *If $Ax = b$ has atmost one solution for any b, f is injective or One to One.*

Definition 6.6. *If a fucntion is both injective and surjective, it is bijective.*

6.5 Kernel or Null Space

Definition 6.7.

$$\{x \mid Ax = 0\}$$

is known as the kernel or null space of A.

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7.1 Recall

If A is a $r \times c$ matrix,

$$f_A : \mathbb{R}^c \to \mathbb{R}^r$$
$$x \to Ax$$

Observe that:

$$A\left(\begin{bmatrix}1\\0\\\vdots\\0\end{bmatrix}\right) = \begin{bmatrix}a_{11}\\a_{21}\\\vdots\\a_{r1}\end{bmatrix}$$

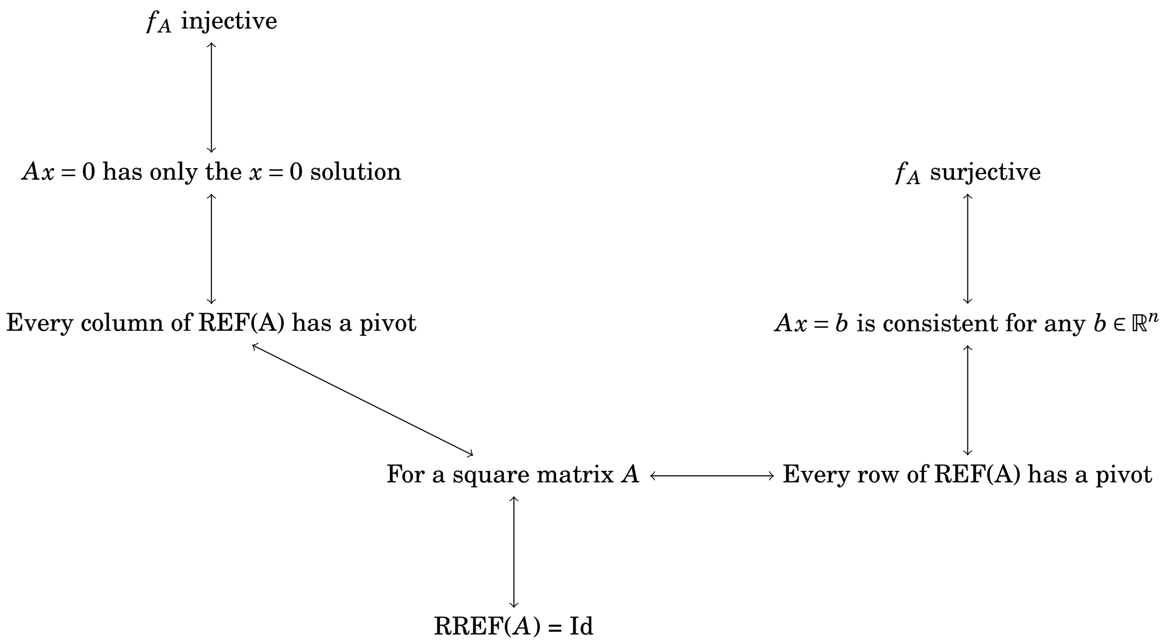
Similarly,

$$A\left(\begin{bmatrix}0\\0\\\vdots\\1\\\vdots\\0\end{bmatrix}\right) = \begin{bmatrix}a_{1i}\\a_{2i}\\\vdots\\a_{ri}\end{bmatrix}$$

Where we have the 1 in the i^{th} column. These type of vectors are called e_i.
Notice that Ae_i is equal to the i^{th} column of matrix A.

Remark 7.1. *Thus, it suffices to know Ae_i for all $1 \le i \le c$ to know the matrix A.*

Remark 7.2. *f_A is injective $\implies c \le r$*
f_A is surjective $\implies r \le c$
f_A is bijective $\implies c = r$
However, these are not if and only if type statements. For r = c, there is no guarentee that f_a is bijective.



7.2 Inverse, for real this time

For a square matrix A,elementary maticies convert A to it's RREF that is I $E_n, \dots, E_2, E_1$.
Notice

$$\underbrace{E_n \dots E_2 E_1}_{E} A = I$$

$$\implies EA = I$$

How do we invert A? Simple, we remember all the row operations we used and multiply them.

Definition 7.3. *A matrix A is said to be invertible if there is a matrix B s.t. $AB = I$ and $BA = I$.*

This clearly implies $A = r \times c$ and $B = c \times r$ then $AB = I_{r\times r}$ and $BA = I_{c\times c}$.

Example 7.4. *(i) If such B exists for A then it is unique. Call it the inverse of A or A^{-1}.*
(ii)

$$(PQ)^{-1} = Q^{-1}P^{-1}$$

Theorem 7.5. *If A is invertible, then $r = c$.*

Proof. Let A, B be matrices such that AB exists.
Then,

$$\begin{aligned} f_a \circ f_b(x) &= f_a \circ Bx \\ &= (AB)x \\ &= f_{AB}x \end{aligned}$$

This implies that $f_A \circ f_B = f_{AB}$.
$AB = I \implies f_A \circ f_B = f_I$
$BA = I \implies f_B \circ f_A = f_I$ Notice that this implies f_A is both injective($r \geq c$) and surjective($r \leq c$) thus, f_A is bijective and $r = c$.
Thus, inverse only exists for square matrices. □

What we just showed is that A is invertible $\implies f_A$ is bijective $\iff r = c$ and $RREF(A) = I$.

Example 7.6. *Prove the converse that is prove that if f_A is bijective, then its unique inverse function is in fact of the form f_B for some B.*

Proof. TBD □

Example 7.7. *For square matrices A, B show that $AB = I \implies BA = I$*

Proof. TBD □

Example 7.8. *Find an example such that $AB = I$ but $BA \neq I$ for A, B not being square matrices.*

Proof. TBD □

Note: Range of f_A is $\mathbb{R}^c$ where c is the number of columns in A.
If

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, Ax = x_1[a_1] + x_2[a_2] + \ldots x_c[a_c] = x_1Ae_1 + x_2Ae_2 + \ldots x_cAe_c$$

7.3 Vector Spaces

Definition 7.9. *A vector space V over $\mathbb{R}$ is a set V with the two opertions*

$$+ : V \times V \to V$$

$$\cdot : \mathbb{R} \times V \to V$$

1. *(1) "V is an abelian group under +" which simply means*
   - $\exists \vec{0} \in V$ s.t. $\vec{v} + \vec{0} = \vec{v} \forall v \in V$
   - $\exists -\vec{v} \in V$ s.t. $\vec{v} + (-\vec{v}) = \vec{0}$
   - $1 \cdot \vec{v} = v$

- $(a+b)\cdot v = a\cdot v + b\cdot v$
- $c\cdot(v+w) = c\cdot v + c\dot{w}$
- $(ab)\cdot v = a\cdot(b\cdot v)$

Here are some exercises:

Example 7.10. *Prove that* 0 *is unique.*

Proof. For the sake of contradiction, assume that there exist more then one zero, let's name two of them $0_1, 0_2$, such that $v + 0_n = v \forall v \in V$.
Note that for this to be true, we need to have $0_1 \neq 0_2$.

$$v = v$$
$$v + 0_1 = v = v = v + 0_2$$
$$v + 0_1 = v + 0_2$$
$$0_1 = 0_2$$

And we have a contradiction!
Thus, our initial assumption is false and hence, only one unique 0 exists. □

Example 7.11. *Prove that the additive inverse of a* $v \in V$ *is unique.*

Proof. For the sake of contradiction, assume that for some $v \in V$ we have more than one additive inverse. Let two of these additive inverses be u and w. Note that this implies $u \neq w$.

$$0 = 0$$
$$v + u = 0 = 0 = v + w$$
$$v + u = v + w$$
$$u = w$$

And we have a contradiction!
Thus, our initial assumption must have been false. Thus, a $v \in V$ has one, unique additive inverse. □

Example 7.12. *Prove that* $0v = 0 \forall v \in V$

Proof.

$$\begin{aligned} &0v \\ &\quad = (0+0)v \\ &\quad = 0v + 0v \\ \implies &0v = 0v + 0v \\ 0v + (-0v) &= 0v + 0v + (-0v) \\ 0 &= 0v \end{aligned}$$

□

Example 7.13. *Prove that* $-v = (-1)v$

$$(-1)v = (-1)v$$
$$(-1)v + 1v = (-1)v + 1v$$
$$(-1+1)v = (-1)v + v$$
$$0v = (-1)v + v$$
$$0 = (-1)v + v$$
$$-v = (-1)v$$

7.4 Subspace

Definition 7.14. *A subpspace W of V is a subset that is closed under the same $+$ and $\cdot$ as V.*
$w_1, w_2 \in W \implies w_1 + w_2 \in W$ $c \in \mathbb{R}, w \in W \implies c \cdot w \in W$ *W itself if a vector space under $+$ and $\cdot$.*

We will now look at some sample vector spaces and subspace.

7.4.1 This is a obvious, but

For $\mathbb{R}^n$ is the only family of examples of vector spaces. Subspaces of $\mathbb{R}^n$ always includes $\{0\}$ and $\mathbb{R}^n$.

7.4.2 The Cartesian plane

For $\mathbb{R}^2$, subspaces include $0, \mathbb{R}^2$. Another one is for some vector $\vec{w}$, the space defined by $r\vec{w}$ where $r \in \mathbb{R}$.
We claim these are the only subspaces. That is if a subspace $S \subseteq \mathbb{R}^2$ is such that $v, w \in S$ and $v \neq kw$ where $k \in \mathbb{R}$, then $S = \mathbb{R}^2$.

Example 7.15. *Prove the above.*

Proof. TBD □

Example 7.16. *Find all subspaces of $\mathbb{R}^n$*

Proof. TBD □

Example 7.17. *Prove that the image of f is a subspace of $\mathbb{R}^n$.*

Proof. TBD □

Example 7.18. *Prove that the kernel of f is a subspace of $\mathbb{R}^n$.*

Proof. TBD □

8 27 August, 2024

8.1 Recall

A was an $r \times c$ matrix which gave us a function of sets $f_A : \mathbb{R}^c \to \mathbb{R}^r$ or $x \to Ax$ where x is a column vector.
Two associated subspaces are $\ker A$ which is also the null space of A and $\text{Im}(A)$ which is basically the range of A.
If $b_1, b_2 \in \text{Im}(A)$ implies $Ax_1 = b_1$ and $Ax_2 = b_2$, which means $Ax_1 + Ax_2 = A(x_1 + x_2) \in \text{Im}(A)$.
If $b \in \text{Im}(A), c \in \mathbb{R}$ which implies $cb = cAx = A(cx) \in \text{Im}(A)$.

8.2 Linear Combinations

Definition 8.1. *In any vector space V,*
let $S = \{v_1, v_2, \dots, v_n\}$ be a (multi) set [5] of vectors.
A linear combination of $v_1, v_2, \dots, v_n$ is any vector of the form $x_1v_1 + x_2v_2 + \cdots + x_nv_n$ where $x_i \in \mathbb{R}$.

Definition 8.2. *Span of $v_1, v_2, \dots, v_n$ is the set of all linear combinations of $v_1, v_2, \dots, v_n$. We can also write this as*

$$\sum_{i=1}^{n} c_i v_i \mid c_i \in \mathbb{R}$$

Note that the span is the subspaces of V where V is the vector space from which the vectors defining the span are drawn.

Example 8.3. *Prove that if a subspace $W \subset V$ contains S then W contains span of S.*

[5] This is a set, but for the sake of argument...

Proof. As W is a subspace, for $v_i, v_j \in W$, $c_i v_i + c_j v_j \in W$ where $c_i, c_j \in \mathbb{F}$ where $\mathbb{F}$ is the field on which W is defined.
If a set of vectors $S = \{v_1, v_2, \ldots, v_n\}$ is contained in W, it is trivial by above that

$$\sum_{i=1}^{n} c_i v_i \in W$$

for $c_i \in \mathbb{F}$.
We remmeber that the definition of span is $\sum_{i=1}^{n} c_i v_i$ whrer $c_i \in \mathbb{F}$ where $\mathbb{F}$ is the field on which the vector space which v_i belong to.
Thus,if a subspace $W \subset V$ contains S then W contains span of S.
And we are done! □

Example 8.4. *For a matrix A, range of f_A which is equal to* $\mathrm{Im}(A)$ *is also equal to the span of column vectors of A.*

Proof. Let $A = \begin{bmatrix} A_1 & A_2 & \ldots & A_c \end{bmatrix}$ where A_i are the column vectors.

$$Ax = \begin{bmatrix} A_1 & A_2 & \ldots & A_c \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = A_1 x_1 + A_2 x_2 + \cdots + A_n x_n$$

which is a linear combination of $A_1, A_2, \ldots, A_n$ and as $x_i \in \mathbb{R}$, we get the range of f_A to be equal to column vectors of A. □

We also need to note that f_A being surjective $\iff$ Span of column vectors of $A = \mathbb{R}^r$ or "columns of A span $\mathbb{R}^r$".
In the same vein, f_A is injective $\iff Ax = 0 \implies x = 0 \iff x_1 A_1 + \cdots + x_c A_c = \vec{0} \implies \forall i, x_i = 0$.

8.3 Linear Independence

Definition 8.5. $v_1, \ldots, v_n \in \mathbb{R}^r$ *are said to be linearly independent, if*

$$\sum_{i=1}^{n} x_i v_i = 0 \implies \forall i, x_i = 0$$

Definition 8.6. *If we can find x_i not all equal to 0 such that $\sum_{i=1}^{n} x_i v_i = 0$, the system $v_1, \ldots, v_n$ is called linearly dependent.*

We can make a theorem here:

Theorem 8.7. *If $v_1, v_2, \ldots, v_n$ are linearly dependent $\iff$ One of $v_1, v_2, \ldots, v_n$ can be written as a linear combination of others.*

Proof. We first prove the $\implies$ part. FTSOC, let none of $v_1, v_2, \ldots, v_n$ be linear combination of others.

$$x_1 v_1 + x_2 v_2 + \ldots x_n v_n = 0$$

We have some $x_i \neq 0$ as the system is linearly dependent. Thus, we can write the above as:

$$x_i v_i = -(x_1 v_1 + x_2 v_2 + \cdots + x_h v_h + x_j h_j + \ldots x_n v_n$$

$$v_i = -\left(\frac{x_1}{x_i} v_1 + \cdots + \frac{x_n}{x_i} v_n\right)$$

The last step works as x_i is non zero.
We now proves the $\impliedby$ part. Let $v_n = x_1 v_1 + \cdots + x_i + v_i$.
Then $x_1 v_1 + \cdots + x_i + v_i - v_n = 0$.
And as both the directions work, Thus, If $v_1, v_2, \ldots, v_n$ are linearly dependent $\iff$ One of $v_1, v_2, \ldots, v_n$ can be written as a linear combination of others. □

8.4 Polynomials as vector spaces

We can say all real polynomials with degree $\leq n$ is a vecttor space in $\mathbb{R}^n$. The zero polynomial is the 0 element. The degree of the 0 polynomial is whatever we want it to be, based on the logic of the space we are working with.
Also note that homogenous polynomials of degree d in k variables is eqavalaent to polynomials in K variables whose every term has the same degree d.

$$\{ax^2 + bxy + cy^2 \mid a, b, c \in \mathbb{R}\} \subset_{\text{subspace}} \{\text{All polynomials in } x \text{ and } y\}$$

Example 8.8. *Polynomials of degree $\leq n$ is the span of* $\{1, x, x^2, \ldots, x^n\}$.

This is clearly tautological.

9 29 August, 2024

9.1 Recap

Vector space of polynomials of degree $\leq n$ is the subspace of $\{1, x, x^2, \ldots, x^n\}$.
Ah, a much shorter recap this time!

9.2 Some trivial facts about linear independence

A trivial fact to note is that any vectors containing $\vec{0}$ is linearly dependent as $0\vec{v_1} + 0\vec{v_2} + \cdots = 0$.
Another trivial fact is any single non-zero vector is linearly independent. That is $c\dot{v} = \vec{0}$ and $v \neq 0$, $\implies c = 0$.
An even more trivial fact is that any multiset in which a vector repeats is linearlt dependent. This is true as if $v_1 = v_2 = v$ and $S = \{v_1, v_2, \ldots\}$ but $v_1 - v_2 = \vec{0}$ is a linear combiniation equalling zero where all coefficients are not zero!
This is similer to a rather trivial fact that $\{v_1 \neq \lambda v_2\}$ is linearly independent. $v_1 = \lambda v_2$ is linearly dependent as $v_1 - \lambda v_2 = \vec{0}$.

9.3 Some trivial facts about span

It is trivial that if $V = \text{Span}(S)$ then any set $S' \supset S$ also spans V.
Another trivial fact is that if $V : \{v_1, v_2, \ldots\}$ is linearly independent then some $v \subset V$ is also linearly independent.

9.4 A lemma!

Lemma 9.1. *Any set $S \in \mathbb{R}^n$ whose span equals $\mathbb{R}^n$ must be of size at least n.*

Proof. Let $S = \{v_i\}_{i=1}^k$.

$$v_i = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$

then consider the matrix

$$A_{n\times k} = \begin{bmatrix} v_1 & v_2 & \ldots & v_k \end{bmatrix}$$

From the hypotheseis, we know that f_A is surjective. This implies that $r \leq c$ or in this case $n \leq k$, which implies that the size of matrix must be atleast n.
And we are done! □

Lemma 9.2. *Any linearly independent $V = \{v_1, v_2 \ldots\}$ on $\mathbb{R}^n$ must have size less than equal to n.*

Proof. This is rather elementry to prove. FTSOC, let there be $m > n$ linearly independent vectors. Let's choose to write some $n+1$ of these vectors as a matrix of the form

$$\begin{bmatrix} v_1 & v_2 & \dots & v_n & | & v_{n+1} \end{bmatrix}$$

Notice that on performing row reduction on this matrix, using the fact $v_1, v_2, \dots, v_n$ are linearly independent, we will have

$$\begin{bmatrix} 1 & 0 & \dots & 0 & | & v_{n+1,1} \\ 0 & 1 & \dots & 0 & | & v_{n+1,2} \\ \vdots & \vdots & \vdots & \vdots & | & \vdots \\ 0 & 0 & \dots & 1 & | & v_{n+1,n} \end{bmatrix}$$

This means $v_{n+1,1}v_1 + v_{n+1,2}v_2 + \dots + v_{n+1,n}v_n = v_{n+1}$.
This contradicts the fact $v_1, v_2, \dots, v_{n+1}$ are linearly independent.
Thus, our initial assumption must be false.
Thus, Any linearly independent $V = \{v_1, v_2 \dots\}$ on $\mathbb{R}^n$ must have size less than equal to n.
And we are done! □

9.5 Basis

Definition 9.3. *A basis of a vector space V is a linearly independent set that spans V*

Example 9.4.

$$e_1 = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

where 1 *is in the* i^{th} *position.*
We claim that $\{e_1, e_2 \dots, e_n\}$ *is a basi for* $\mathbb{R}^n$.

Proof. **Spans?**

$$\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = c_1e_1 + c_2e_2 + \dots + c_ne_n$$

Linearly Independent?

$$\sum c_ie_i = \vec{0} \implies \forall i, c_i \neq 0$$

□

Theorem 9.5. *Any basis of* $\mathbb{R}^n$ *must contain exactly n elements. That is "dimension of* $\mathbb{R}^n$*" is n.*

9.6 Other bases for $\mathbb{R}^n$

1. $\mathbb{R}^1$: Any non-zero element is a basis.
2. $\mathbb{R}^2$: $\{u, v\}$ where u, v are non-zero vectors not lying on the same line through the origin.
3. $\mathbb{R}^3$: $\{u, v, w\}$ where u, v, w non zero vector not lying on a plane through the origin.
4. Basis for a polynomial of degree $\leq n$ is

$$\{1, x, x^2, \dots, x^n\}$$

$c_0 + c_1x + c_2x^2 + \dots + c_nx^n \equiv 0 \implies \forall c_i = 0$ but we can also write this as $\{1, (x-1), (x-1)^2, \dots, (x-1)^n\}$.

Also note that polynomials of degree $\leq n$ is a vector space of $\mathbb{R}^{n+1}$ as the constant term also accounts for a dimension.

9.7 A question

Example 9.6. *Does a general vector space V have a basis?*

Proof. **Strategy A** Start with a empty set and keep adding vectors till they span.

Lemma 9.7. *Suppose $S \subset V$, S is linearly independent and u is such that $u \notin \text{Span}(S)$ then $S \cup \{u\}$ is also linearly independent.*

Proof.

$$\sum_{v_i \in S} c_i v_i + \lambda u = 0$$

should imply $c_i, \lambda = 0$
If $\lambda = 0$, then $\sum_{v_i \in S} c_i v_i = 0 \implies c_i = 0$
If $\lambda \neq 0$, then

$$u = \sum_{v_i \in S} \left(\frac{-c_i}{\lambda} v_i \right)$$

which is absurd as u couldn't be written as a linear combination $V = \{v_1, v_2, \ldots\}$. □

Problem: How do we know that this terminates?
Strategy B Start with a spanning set and we remove elements till it is linearly independent.

Lemma 9.8. *Suppose $S \subset V$. Suppose $v \in S$ and $v \in \text{Span}(S \setminus \{v\})$ such that $\text{Span}(S) = \text{Span}(S \setminus \{v\})$*

Problem: We are not sure whether this terminates. □

Definition 9.9. *V is called a finite dimensional vector space if and only if it has the finite spanning set.*

Theorem 9.10. *Every finite dimensional vector space has a basis.*

Theorem 9.11. *Every basis of a finite dimensional vector space has the same cardinality.*

10 10 September, 2024

10.1 Recall

Definition 10.1. *V is called a finite dimension vector space(fdvsp) if it has a finite spanning set.*

Theorem 10.2. *Every fdsvp has a basis.*

This is trivial.

Theorem 10.3. *Any two bases have the same cardinality.*

Example 10.4. *For $\mathbb{R}^n$, any basis has n vectors.*

10.2 The Abstract proof of this

Lemma 10.5 (Steinz Exchange Lemma)**.** *Suppose $v \in S \subset V$ and $u \in V$ s.t. $u \in \text{Span}(S)$. Suppose $u \notin \text{Span}(S \setminus \{v\})$, then $v \in \text{Span}(S \setminus \{v\} \cup \{u\})$.*

Proof.

$$S = \{v, v_2, \ldots, v_k\}$$

$$u = av + \sum_{i=2}^{k} a_i v_i$$

The above holds since $u \in \text{Span}(S)$.
$a \neq 0$ because $u \notin \text{Span}(S \setminus \{v\})$.

$$\implies \left(\frac{1}{a}\right) u - \sum_{i=2}^{k} \left(\frac{a_i}{a}\right) v_i = v$$

□

Proposition 10.6. *If I is a finite linearly independent set and S is any spanning set, then $|I| \leq |S|$*

Proof.

$$I = \{u_1, u_2 \dots u_m\}$$

$$S = \{v_1, v_2 \dots v_n\}$$

To Show: $m \leq n$ and after rearranging og v_i if necessary, $\{u_1, u_2, \dots, u_m, v_{m+1}, \dots, v_n\}$ spans V.

Proof. Inducting on m.
(B) for $m = 0$, holds as there are no $u_i's$ and S spans V.
(S) Assume that the reuslt holds for $|I| = m-1$, in particuler $m - 1 \leq n$.

$$S' = \{u_1, \dots, u_{m-1}, v_m, v_{m+1} \dots v_n\}$$

spans V.

$$u_m = \sum_{i=1}^{m-1} c_i u_i + \sum_{j=m}^{n} d_j v_j$$

Observe $d_j \neq 0$, say d_m after possible rearranging te $v_j's$. iN particuler, $m \leq n$.
Further, $u_m \in \text{Span}\left(S'\right), u_, \notin \text{Span}\left(S' \setminus \{v_m\}\right)$.
Exchange v_m by u_m and we are done! ☐

Thus, the proposition implies for fdvsp V, cardinality of any two bases is the same. ☐

10.3 Stratergy to Build a Basis

Stratergy A is we start with a linearly independent set and add vectors till it is spanning.

Proof.

$$I_0 = I, S = \text{spanning set}$$

$$I_1 =$$

if $u_1 \notin \text{Span}(I)$, then $I_1 = I_0 \cup \{u_1\}$.
of $u_i \in \text{Span}(I)$, then $I_1 = I_0$. Observe : I_1 is linearly independent.
Form I_j iterativly. Each I_j is linearly independent and $\text{Span}(I_k) = V$.
Thus, I_k is a basis. ☐

Stratergy B is we start with a spanning set and delete vectors till it is linearly independent.

Proof. TBD ☐

10.4 Another proof using isomorphism

Observe: A basis defines a coordinare system on fdvsp V.
How? Any $u \in V$ can be written uniquey as a linear combination of basis vectors.

$$u = \sum_{v_i \in B} c_i v_i$$

where $B = \{v_1, v_2, \dots, v_n\}$ is some given basis.
The uniquness is simply proved by subtracting and is not included here for brevity.
If we fix the order in which $B = \{v_1, v_2 \dots, v_n\}$ is written, then $(c_1, c_2, \dots, c_n)$ is the corrdinate vector in $\mathbb{R}^n$ of u with respect to the ordered basis B.

Definition 10.7. *Suppose V and W are vector spaces. A function $f : V \to W$ is called an isomorphism if:*
(0) *f is a bijection.*
(1) *$f(v_1 + v_2) = f(v_1) + f(v_2)$ and $f(\lambda v_1) = \lambda f(v_1) \forall \lambda \in \mathbb{R}, \forall v_1, v_2, v \in V$.*

Remark 10.8. *If f is an isomorphism, so is f^{-1}.*

$$f^{-1}(w_1 + w_2) =^? f^{-1}(w_1) + f^{-1}(w_2)$$

$\implies w_1 = f(v_1)$ *for some unique* $v_1 \in V$.
$\implies w_2 = f(v_2)$ *for some unique* $v_2 \in V$.
$\therefore w_1 + w_2 = f(v_1) + f(v_2) = f(v_1 + v_2)$
$\therefore f^{-1}(w_1 + w_2) = v_1 + v_2 = f^{-1}(w_1) + f^{-1}(w_2)$.
Similer for scaler multiplication.

Remark 10.9. *If there is an isomorphism from $V \to W$, we say V and W are isomorphic vector spaces.*

Remark 10.10. *If $f : \mathbb{R}^c \to \mathbb{R}^r$ given by $x \to Ax$ for some some $r \times c$ matrix A then f satisfies* (1) *by defination.*

f is an isomorphism $\iff RREF(A) = Id$ so $r = c$.
$\iff$ A in invertible.
If $f : v \to w$ is an isomorphsim then any true assertion in V about this vector space structure remains true after transport by f and vice versa for transport by f^{-1}.

Example 10.11. *S is a basis of $V \iff f(S)$ is a basis of W.*

Proof of theorem (2). V fdvsp.
Let $\{v_1, v_2, \ldots, v_n\}$ be a basis.
$\exists$ an isomorphism $f : \mathbb{R}^n \to V$.

$$\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} \to \sum c_i v_i$$

If $\{u_1, \ldots, u_k\}$ is another basis.
Then $\exists$ isomorphism $g : \mathbb{R}^k \to V$.

$$\begin{bmatrix} d_1 \\ \vdots \\ d_n \end{bmatrix} \to \sum d_j v_j$$

$g^{-1} \circ f : \mathbb{R}^n \to \mathbb{R}^k$ is an isomorphism.
$\implies n = k$ □

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11.1 Say hi to Isomorphisms!

(i) $f : V \to W \text{isom} \implies f^{-1} : W \to V$. f is an isomorphism $V \cong W$ or "V is isomorphic to W".
(ii) $f : V \to W\text{isom}, g : W \to U\text{isom} \implies f \circ f : V \to U$ is an isomorphism. We can write this also as $V \cong W, W \cong U \implies V \cong U$.
(iii) If S is a basis for V then $f(S)$ is a basis for W.
(iv) If $(v_1, v_2, \ldots, v_n$ is an ordered basis then we get $f : \mathbb{R}^n \to V$ is an isomorphism.

$$f\left(\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}\right) = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n$$

Theorem 11.1 (Invarience of Dimension)**.** *Let V be a fdvsp. Then, any two basis of V have the same (neccesarily finite) cardinality. This is called the **dimension** of V (over $\mathbb{R}$). In this case, $V \cong \mathbb{R}^n$, if $|basis| = n$.*

11.1.1 Corollaries

(i) If S is any spanning set with $|S| = n$, the in fact, S is linear independent and hence a basis.
(ii) If I is any linear independent set with $|I| - n$, then in fact, I is spanning set, and hence, a basis.
(iii) If $W \subseteq V$ is a subspace, then $\dim W \leq \dim V$.

Proof. Take a basis for W. W is finite dimensional because spanning set for V will give a spanning set for W.
B stays linearly independent over V. Then, by a previous proposition,

$$\lim W = |B| \leq |\text{basis for } V| - \lim V$$

□

Remark 11.2. *For vector spaces that are not fdvsp. Existence of a basis can be shown using stratergy A and Zorn's Lemma/Axiom of choice.*

Remark 11.3. *Uniqueness of cardinalities also holds.*

11.2 How to calculate with Basis

Example 11.4. *Does* $\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}3\\8\end{bmatrix}$ *form a basis of* $\mathbb{R}^2$

Solution. Check linear indepndence, or try to create $\begin{bmatrix}1\\0\\\vdots\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\\vdots\\0\end{bmatrix}, \ldots, \begin{bmatrix}0\\\vdots\\0\\1\end{bmatrix}$. □

Example 11.5. *Prove that* $\{1, x, x^2\}$ *is a basis of polynomials of degree* ≤ 2. *What about* $\{1, 1+x, 1+x+x^2\}$*?*

Example 11.6. *A* $r \times c$ *matrix,* $f_A : \mathbb{R}^c -> \mathbb{R}^r$ *or* $x \to Ax$.
How to find basis of:
(i) $\ker f_A = \text{nul}(A)$
(ii) $\text{Im}(f)_A = \text{Span}(\textit{column of } A)$

12 12 September, 2024

12.1 Recall

$$R = \begin{bmatrix} 0 & 1 & P & 0 & Q & 0 \\ 0 & 0 & 0 & 1 & r & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$RX = 0$$

has the general solution

$$X_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + X_3 \begin{bmatrix} 1 \\ -p \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + X_5 \begin{bmatrix} 0 \\ -q \\ 0 \\ -r \\ 1 \\ 0 \end{bmatrix}$$

That is

$$\text{nul}(A) = \text{Span}(V_1, V_3, V_5)$$

and throws corresponding to the indecies 1,3,5 imply that they are linearly independent.

$$\because \text{A basis for Nul}(A) = \{v_1, v_3, v_5\}$$

Conclusion: Dimension of null space is equal to number of free variables.

12.2 Basis of $\text{Im}(A) = \text{col}(A)$

Stratergy (i) Make rows into columns.

Definition 12.1. *The transpose of N a $p \times q$ matrix, M^t or transpose of M is a $q \times p$ matrix givent by $(M^t)_{ij} := M_{ji}$.*

12.3 Some easy properties of transpose

Theorem 12.2.

$$(M^t)^t = M$$

$$(M+N)^t = M^t + N^t$$

$$(cM)^t = cM^t$$

For A a $p \times q$ matrix and B a $q \times r$ matrix. Notice, A^t is $q \times p$ and B^t is $r \times q$

Theorem 12.3.

$$(AB)^t = B^t A^t$$

Proof. TBD □

Theorem 12.4.

$$(AB)^{-1} = B^{-1}A^{-1}$$

for square matrices.

Also note $A \to (A^t)^{-1}$ is a intresting function to think about

12.4 To column operations

Notice,

$$Ax = B \equiv x^t A^t = B^t$$

where x and B are row vectors.
We can solve such equations using column operations.
Doing a column operation is same as multiplying on the right by an elementry matrix.

$$A \to AE$$

Such matrices can be converted to reduced column echelon form, which is just transposed form of RREF.

12.5 Actually the basis for $\text{Im}(f_A) = \text{Col}(A) = \text{Span}($Columns of $A)$

Note that $\text{Col}(A) \subseteq \mathbb{R}^r$.
Similerly, what is $\text{Row}(A) := \text{Span}(\text{Rows of } A)$.
Also note, $\text{Row}(A) \subseteq \mathbb{R}^c$.
Case 1 A is an RREF.
The pivot columns are linearly independent another columns are in their span.
In this case: A basis of $\text{Col}(A) = \{\text{Pivot columns of } A\}$.
General Case: Row operations change the column space of A.

Example 12.5.

$$\begin{bmatrix}\sqrt{2}\\ \pi\end{bmatrix} \xrightarrow{\textit{Row operations}} \begin{bmatrix}1\\ 0\end{bmatrix}$$

But notice that the span of both the forms is not the same.

Insight: Find linear relations among column vectors of A, so that we can drop "unnecessary" columns. This is same as finding non-trivial solution to

$$Ax = 0$$

$\iff$ Finding non-trivail solutions to

$$\text{RREF}(A)x = 0$$

Any set of columns of RREF(A) is linearly dependent/independent $\iff$ corresponding set of columns of A is linearly dependent or independent.
$\because$ A basis for $\text{Col}(A) = \{$Columns of A corresponding to the pivot columns of $RREF(A)\}$.

Remark 12.6. *Nullity of A = Dimension of null space = Number of free variables*
(Column)[6] *rank of A = Dimensin of* $\text{Col}(A)$ *= Number of pivot variables.*

Theorem 12.7 (Rank Nullity Theorem)**.** *(Column) Rank of A + Nullity of A = Number of columns of A*

We can also write it as

Theorem 12.8.

$$\dim \text{Im}(f_A) + \dim \ker f_A = \dim \mathbb{R}^c$$

where $\mathbb{R}^c$ *is the domain of* f_A.

12.6 Basis for Row(A) = span of row vectors of A

Case 1: If A is RREF, then the non-zero rows form a basis as thet are linearly independent.
General Case: Arbitry Matrix $A \xrightarrow{\text{Row Operations}} A'$, then Rows of A' are linear combinations of rows of A and vice versea.
A basis of $\text{Row}(A) = \{$Nonzero row vectors of $RREF(A)\}$
This implies

$$\dim \text{row} A = \text{Number of pivots in } RREF(A) = \dim \text{col} A$$

Thus, column rank is equal to row rank.[7]

Theorem 12.9. *Rank of A = Column rank of A = Row rank of A*

12.7 Back to abstract vector spaces

Example 12.10. *Set of all* $p \times q$ *matrices is a vector space*[8] *under entry-wise addition and scalar multiplication.*

Example 12.11. *Show that V is isomorphic to* $\mathbb{R}^{pq}$

Proof. TBD □

Example 12.12. *If* X *is any set,* $W = \{f : X \to \mathbb{R}\}$ *is a vector space under pointwise addition and scaler multiplication.*

Example 12.13. *If X is finite, then what is* $\dim W$*?*

Proof. TBD □

[6] Eventually, we will find column rank is equal to row rank
[7] Told ya!
[8] As we can also multiply matrices, we can upgrade this from a vector space to an algebra. But we study that in algebra III

13 17 September, 2024

13.1 Recall

An $r \times c$ matrix $A \rightsquigarrow f_A : \mathbb{R}^c \to \mathbb{R}^r$ $f_A(x) = Ax$
$\ker(f_A) \subset \mathbb{R}^c$
$\mathrm{Im}(()f_A) \subset \mathbb{R}^r$
$\dim(\ker f_A) =$ number of non-pivot colums in $RREF(A)$.
$\dim(\mathrm{Im}(f)_A) =$ number of pivot colums in $RREF(A)$.
We also proved $\dim(\ker f_A) + \dim(\mathrm{Im}(f)_A) = \dim($domain of f_A
The former is also called the nullity and the latter nullity, providing the name, Rank-Nullity Theorem.

13.2 Linear Maps

Definition 13.1. *A map $T : V \to W$ is called "linear" if it satisfies (1) $T(V_1 + V_2) = TV_1 + TV_2 \forall v_1, V_2 \in V$ (2) $T(cV) = cT(V) \forall c \in \mathbb{R}, \forall v \in V$.*

The goal:

$$\dim \ker T + \dim \mathrm{Im}(T) = \dim V$$

Here kernal refers to a subset of the comdomain of T such that $\{x|T(X) = 0\}$. Image is just the range.

Remark 13.2. *(I) A linear map T is a map that preserves the vector space structure.*
(II) Adding first in V and then applying T is the same as applying T first and then adding in W.
Also note

$$T(C_1v_1 + C_2v_2) = T(C_1v_1) + T(C_2v_2) = C_1T(v_1) + C_2T(v_2) \implies T\left(\sum_{i=1}^{n} C_iv_i\right) = \sum_{i=1}^{n} C_iT(v_i)$$

(III) For a linear map $T : V \to W$, (i) $T(\vec{0_v}) = T(0 \cdot \vec{0_v}) = 0T(\vec{0_v}) = \vec{0_W}$
(ii) $\ker T = \{v \in V | T(v) = 0\}$ is a subspace of V.
(iii) $\mathrm{Im}(T) = \{T(v) | v \in V\}$ is a subspace of W.
(iv) If $S \in V$ spans V, then $T(S)$ spans $T(V) \equiv \mathrm{Im}(T)$
(v) If $S \in V$ is linearly dependent then $T(S) \in T(V)$ is linearly dependent.

13.3 Some examples

Example 13.3. *For $r \times c$ matrix A, $f_A : \mathbb{R}^c \to \mathbb{R}^r$ or $x \to Ax$ is linear.*

Here is a bit more involved example

Example 13.4. *V = set of all diffrentiable functions froms $\mathbb{R} \to \mathbb{R}$.*
W = set of all functions from $\mathbb{R} \to \mathbb{R}$.
Define $D : V \to W$ or $f \to f'$.
This is clearlt a linear map as: (i) (f+g)' = f' + g'
(ii) (cf)' = cf'
(iii) $D = c$ where c is constent.

Example 13.5. *Consider the same map on the subspace V of polynomials of degree $\leq n$.*
Consider the same map on the subspace V of polynomials of degree $\leq n$

$$\dim(\ker D) + \dim(\mathrm{Im} D) = \dim(\mathrm{Domain} D)$$

$$P_n \xrightarrow{D} P_{n-1}$$

$$\dim = n+1 \qquad \dim = n$$

$$\dim \textit{of} \ker D = 1$$

[9]

Example 13.6. *V = set of $n \times n$ matrices.*
$W = \mathbb{R}$
$T : V \to W$ *such that* $A \to \text{trace}(A)$.
Check that the dimension,formula holds and find a basis for the kernel.

Solution. TBD □

13.4 Rank-Nullity abstracted

Theorem 13.7. *For a linear map $T : V \to W$ where V is finite dimensional vector space, then we have:*

$$\dim \text{Im}(T) + \dim \ker T = \dim V$$

Proof. **We will show:** (Basis of $\ker T$) $\cup$ (A disjoint set of size $\dim \text{Im}(T)$) = (A basis of V). (1) Take a basis $k_1, k_2, \ldots, k_t$ of $\ker T$.
(2) Extend to get a basis $k_1, k_2, \ldots, k_2, v_1, \ldots, v_r$ of V.
$\dim \ker T = t$, $\dim V = t + r$,
To show: $\dim \text{Im}(T) = r$.
Guess $T(v_1), T(v_2) \ldots T(v_r)$ is a basis of $\text{Im}(T)$.
(i) **The vectors span** $\text{Im}(T)$
Since $\{k_1, \ldots, k_t, v_1, \ldots, v_r\}$ is a basis of V.
$\{T(k_i)\}$ and $\{T(v_j)\}$ span $T(V) = \text{Im}(T)$.
This is true as $\{0\}$ and $\{T(v_j)\}$ span $T(V) = \text{Im}(T)$.
(ii) $T(v_1), \ldots, T(v_r)$ are linearly independent.

$$\sum_{i=1}^{r} C_i T(v_i) = 0$$

$$\implies T\left(\sum_{i=1}^{r} C_i v_i\right) = 0$$

As $\sum_{i=1}^{r} C_i v_i \in \ker T$,

$$\sum_{i=1}^{r} C_i v_i = \sum_{j=1}^{t} a_j k_j$$

$$\sum_{i=1}^{r} C_i v_i - \sum_{j=1}^{t} a_j k_j = 0$$

Since $\{k_j\}_{j=1}^{t}$ and $\{v_i\}_{i=1}^{r}$ form a basis. This implies all c_i and all $a_j = 0$. In partuculer, $T(v_1), T(v_2), \ldots, T(v_n)$ are linearly independent.

□

14 19 September, 2024

Last class before midterm. We will have a quiz on 24^{th} and have a preparatory break on 27^{th}.

[9]Sir told us that the ability to let differentiation be linear map is very useful in number theory. He didn't tell us why.

14.1 Recall

$T : V \to W$ is linear map then $\dim\ker T + \dim \mathrm{Im}(T) = \dim V$.
Recipe Take a basis of $\ker T \subset V$.
Extend to a basis of V then corresponding images under T will give a basis for $\mathrm{Im}(T)$.

14.2 The matrix-linear transform bijection

Theorem 14.1. *Any linear map T from $\mathbb{R}^p \to \mathbb{R}^q$ is of the form $f_A(X)$ for some A a $q \times p$ matrix.*

Proof. Take

$$x = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_p \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} + \cdots + c_p \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} = c_1 e_1 + c_2 e_2 + \cdots + c_p e_p$$

$$\therefore T\left(\sum_{i=1}^{p} c_i e_i\right) = \sum_{i=1}^{p} c_i T(e_i)$$

Let

$$T(e_i) = u_i = \begin{bmatrix} u_{i1} \\ u_{i2} \\ \vdots \\ u_{iq} \end{bmatrix}$$

$$\therefore T\left(\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_p \end{bmatrix}\right) = c_1 \begin{bmatrix} u_{11} \\ u_{21} \\ \vdots \\ u_{q1} \end{bmatrix} + c_2 \begin{bmatrix} u_{12} \\ u_{22} \\ \vdots \\ u_{q2} \end{bmatrix} + \cdots + c_p \begin{bmatrix} u_{1p} \\ u_{2p} \\ \vdots \\ u_{qp} \end{bmatrix} = \begin{bmatrix} u_{11} & u_{12} & \dots & u_{1p} \\ u_{21} & u_{22} & \dots & u_{2p} \\ \vdots & \vdots & \dots & \vdots \\ u_{q1} & u_{q2} & \dots & u_{qp} \end{bmatrix}$$

□

The matrix A corresponding to T (ie A such that $T = f_A$) is equal to matrix whose columns are $T(e_1), T(e_2), \dots, T(e_p)$ in that order.

14.3 Some examples

Observe the following maps are linear for geometric reasons. Find there matrices.

Example 14.2.

$$\mathbb{R}^2 \to \mathbb{R}^2, \textit{Rotation by angle } \theta$$

Solution.

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

□

Example 14.3.

$$\mathbb{R}^3 \to \mathbb{R}^3, \textit{Rotation by angle } \theta \textit{ about one of the axis}$$

Solution.

$$R_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

$$R_y = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

$$R_z = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

□

Example 14.4.

$$\mathbb{R}^2 \to \mathbb{R}^2, \textit{Reflection about the line} y = x$$

Solution.

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

□

Example 14.5.

$$\mathbb{R}^3 \to \mathbb{R}^3, \textit{Reflection in the } XY \textit{ plane}$$

Solution.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

□

Example 14.6.

$$\mathbb{R}^2 \to \mathbb{R}^2, \textit{Rotation by angle } \theta \textit{ followed by reflection in the line } y = x$$

Solution.

$$\begin{bmatrix} \sin\theta & \cos\theta \\ \cos\theta & -\sin\theta \end{bmatrix}$$

□

Remark 14.7. *On the choice of basis, we may have many different, isomorphic matrices of linear transform. Choose one basis and stick with it!*

14.4 Recall again,

Matrix of linear map $T : V \to W$ for any finite dimensional V, W by picking a basis for V and a basis for W.
Suppose $v_1, v_2 \dots v_n$ is a list of vectors in a vector space V.
Then, we can define a map

$$\psi : \mathbb{R}^n \to V$$

$$\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} \to c_1 v_1 + c_2 v_2 + \cdots + c_n v_n$$

Observe: (i) ψ is linear.
(ii) ψ is onto $\iff$ $v_1, \dots, v_n$ are linearily independen.
(iii) ψ is injective $\iff$ $v_1, \dots, v_n$ are linearly independent.
(iv) ψ is an isomorphism $\iff$ $v_1, \dots, v_n$ is a basis.

Take a ordered basis $B = (v_1, v_2, \dots, v_n)$ for $n-$dimensional vector space V.
Then Inverse of ψ gives coordinates of any $v \in V$ with respect to given basis B.
That is, if we weite $v = \sum_{i=1}^{n} c_i v_i$, uniquely. The the coordinate vector of v with respect to B is

$$\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \psi^{-1}(v)$$

Let $T : V \to W$ be linear.
Let $v_1, v_2, \dots, v_p$ be an ordered basis for V.

Let $w_1, w_2, \dots, w_q$ be an ordered basis for V.

$$V \to W$$

$$\psi \mathbb{R} \to \mathbb{R} \phi$$

$$\mathbb{R}^p \xrightarrow{\phi^{-1} \circ T \circ \psi} \mathbb{R}^q$$

$$v \to T(v)$$

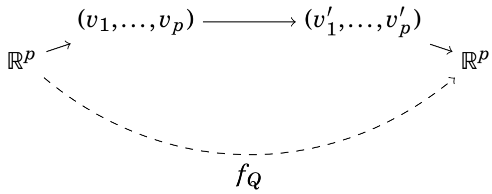
$$\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_p \end{bmatrix} \to \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_p \end{bmatrix}$$

We must have a matrix A s.t.

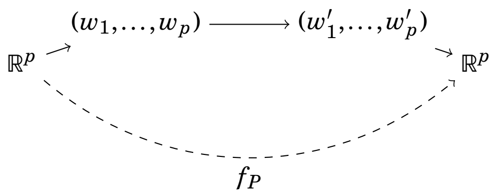
$$A\left(\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_p \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_q \end{bmatrix}\right)$$

The matrix A has the property, its columns are images of the standard basis vectors under the linear map $\phi^{-1} \circ T \circ \psi$.
Answer: Matrix of T with respect to given bases of V and W is obtained as its columns are coordinate vectors of T(given basis of V) with respect to given basis of W.

Theorem 14.8 (Change of Basis Formula). *$(v_1, v_2, \dots, v_p)$ and $(v'_1, v'_2, \dots, v'_p)$ are two ordered basis of V. $(w_1, w_2, \dots, w_q)$ and $(w'_1, w'_2, \dots, w'_q)$ are two ordered basis of W.*
Then if A is the matrix of $T : V \to W$ for the given bases $(v_1, \dots, v_p)$ and $(w_1, \dots, w_q)$.
And we can change bases from $(v_1, \dots, v_p)$ to $(v'_1, \dots, v'_p)$



and similarly



The the matrix of $T : V \to W$ for the other given bases $(v'_1, \dots, v'_p)$ and $(w'_1, \dots, w'_q)$ is

$$PAQ^{-1}$$

15 8 September 2024

15.1 Recall

$T : V \to W$ is a linear map between bases $\beta_V \to \beta_W$.
Matrix of T in given bases is obtained its columns are coordinate vectors of $T(\beta_v)$ with respect to the basis β_W.

15.2 Change of Basis

V space of dim$N$, old basis $\beta_V = (v_1, \dots, v_n)$ and new basis $\beta'_V = (w_1, \dots, w_n)$.

$$W_j = p_{1j}\vec{v_1} + p_{2j}\vec{v_2} + \dots + p_{1m}\vec{v_n}$$
$$= \begin{bmatrix} \vec{V_1} & \vec{V_2} & \dots & \vec{V_n} \end{bmatrix} \begin{bmatrix} p_{1j} \\ p_{2j} \\ \vdots \\ p_{nj} \end{bmatrix}$$

P will be an invertible matrix as going from $V_k \underset{P}{\rightarrow} W_k \underset{P^{-1}}{\longrightarrow} V_k$ as $v_k \underset{id}{\longrightarrow} v_k \implies PP^{=1} = id$.
Any invertible matrix can be taught of as a change of basis matrix.

$$M_{\beta_v,\beta_w}(T) = Q^{-1}AP$$

15.3 Composition of lnear maps in terms of notation.

$$V \underset{T}{\rightarrow} W \underset{S}{\rightarrow} U$$
$$\dim V, W, R = p, q, r$$
$$\text{Basis} V, W, R = \beta_V, \beta_W, \beta_U$$
$$M_{\beta_v,\beta_w}(S \circ T) = M_{\beta_v,\beta_w} S \circ M_{\beta_v,\beta_w} T$$

Proof. Take a look at (2,3) entry of $M_{\beta_v,\beta_w}(S \circ T)$.
= Coeffciecient of u_2 in $(S \circ T)v_3$
Now, $(S \circ T)v_3 = S(\sum_{j=1}^{q} t_{j3} w_j)$.
If matrix of $T = (t_i j)$
If matrix of $S = (s_i j)$

$$(S \circ T)v_3 = \sum_{j=1}^{q} t_{j3} S(w_j) = \sum_{j=1}^{q} t_{j3} \sum_{i=1}^{r} s_{ij} u_i$$

Coeff of $u_2 \leftrightarrow i = 2$
Coeff of $u_2 = \sum_{j=1}^{q} S_{2j} t_{j3} = (2,3)$ entry of matrix product $M(S)$ and $M(T)$. □

15.4 "Good basis for a linear map

$$T : V \to W$$

where rank pf $T = r$.
Take a absis of ker $T : k_1, \dots k_{p-r}$.
Extend to get a basis pf V:

$$k_1, k_2 \dots k_{p-n}, v_1, \dots v_n$$

We know : $T(v_1) \dots T(v_n)$ form a basis of T in W.
Extend this to a bsis of W.

$$T(v_1), T(v_2), \dots, w_1, \dots, w_{q-r}$$

Thus, matrix of T in β_V, β_W

$$\left[\begin{array}{cccc|cccc} 0 & 0 & \dots & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 1 \\ \hline 0 & 0 & \dots & 0 & - & - & - & - \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \end{array}\right]$$

15.5 Direct Sums

Definition 15.1. *External diect sum of W_1, W_2 is $W_1 \oplus W_2$*

Let $W_1 \oplus W_2 = W_1 \times W_2$.
Addition: Slotwise

$$(w_1, w_2) + (w_1' w_2') = (w_1 + w_1', w_2 + w_2')$$

Scaler Multiplication : Slotwise

$$c(w_1, w_2) = (cw_1, cw_2)$$

<To be filled>

16 MIDSEM!!!

17 9 October 2024

17.1 Recall

External direct sum.
W_1, W_2 are vector spaces.
$W_1 \oplus W_2$, as a set it is $W_1 \times W_2$.

17.1.1 Addition

$(w_1, w_2) + (w_1', w_2') = (w_1 + w_1' + w_2 + w_2')$

17.1.2 Scaler Multiplication

$c(w_1, w_2) = (cw_1, cw_2)$ <Include the diagram from last class but with hooked arrows>

17.2 A lemma

Lemma 17.1.

$$\dim(w_1 \oplus w_2) = \dim w_1 + \dim w_2$$

Proof. If $\{v_1\}$ □

<God fucking knows how much is to be completed as the CMOS just had to crash.>
Analogue: A, B are finite sets.

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Example 17.2. *Write analogous form of*

$$W_1 + W_2 + \cdots + W_k$$

Artin's Proof. Take a bssis of $W_1 \cap W_2$. Let's call it $\alpha_1, \alpha_2, \ldots, \alpha_k$.
(i) Extend to a bsis of W_1 by attaching $\beta_1, \beta_2, \ldots, \beta_l$.
(ii) Extend to a basis of W_2 by attaching $\gamma_1, \gamma_2, \ldots, \gamma_m$.
Then claimed formula is $(k+l)+(k+m) = \dim(W_1+W_2)+k$ **To Show:** $\dim(W_1+W_2) = k+l+m$. - $\alpha's, \beta's, \gamma's$ form a basis for $W_1 + W_2$ □

In general,

$$S : W_1 \oplus W_2 \oplus \cdots \oplus W_l \to W_1 + W_2 + \cdots + W_n \subseteq V$$

$$(w_1, w_2, \ldots, w_k) \to \sum_{i=1}^{k} w_i$$

S is surhjective if and oly if $W_1, \ldots, W_k$ span V. S is injective if and only if $W_1, \ldots, W_k$ is linearly independent.

Remark 17.3. *$k = 2$, W_1, W_2 being linearly independent if and only if $W_1 \cap W_2 = \{0\}$*
$k = 3$, w_1, w_2, w_3 are linearly independent implies $w_1 \cap w_2 = \{0\}, w_2 \cap w_3 = \{0\}, w_3 \cap w_1 = \{0\}$.
Note the converse is not true!

17.3 DETERMINANTS!

Determinant is a function.

$$\det : \{n \times n \text{square matrices}\} \to \mathbb{R}$$

[10] satisfying

1. Doing a row replacement $R_i \to R_i + \lambda R_j$ does not change the determinant.

2. Scaling a row by scaler multiple c the determinenant by c.

3. Swaping two rows multiples the determineant by -1.

4. Determinant of $I_n = 1$

17.3.1 Motivation:

Geometric The area of a parallelogon spanned by vectors (a, b) and (c, d) is its determinent.
The area of a parallelopipete spanned by vectors (a, b, c),(d, e, f) and (g, h, i) is its determinent.
If we show these properties hold for all such vectors, then we will show these properties hold as two different volume functions don't give two different volumes.

1. Doing a row replacement $R_i \to R_i + \lambda R_j$ does not change the determinant,

   $$v_1, v_2, \ldots, v_n \text{ are vectors in } \mathbb{R}^n$$

   We can imagine all $v_2, \ldots, v_n$ are compressed into one U_2. Then volume of parallelogram U_2, V_1 is the determinent.
   Switching $V_2 \to V_2 + \lambda v_1$ doesn't change the height and the base is still V_1. So the area doesn't change.

2. Scaling a sice will scale the volume by the same factor.

3. Swapping wo rows does not change volume.

4. Rows of identity matric are standard basis vectors and hence have area 1.

17.4 Computing determinants by Row Reduction

A square matrix A can be row reduced to $REF(A) = B$,
Then

$$\det(A) = (-1)^r \frac{\text{(Product of diagonal entires of } B}{\text{product of scaling factors}}$$

where r is the number of rows swapped.

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18.1 Recipe of computing determinants by row reduction

Claim: A square matrix A can be row reduced to $REF(A) = B$,
Then

$$\det(A) = (-1)^r \frac{\text{(Product of diagonal entires of } B}{\text{product of scaling factors}}$$

where r is the number of rows swapped.
Notice: Any RREF is upward triangular ie RREF is of the form:

$$\begin{bmatrix} * & * & \ldots & * \\ 0 & * & \ldots & * \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & * \end{bmatrix}$$

[10] In general, $\mathbb{R}$ is replaced by whatever field we are dealing with.

basically, $a_ij = 0$ if $i > j$.
To Prove: det(upper/lower triangular matrix = Product of diagonal entries.

Proposition 18.1. *Let A be a $n \times n$ matrix.*

1. *If A has a zero row (or column), then let* $\det A = 0$.

2. *If A is upper / lower triangle the* $\det A$ *is the product of its diagonal entries.*

Proof. (1) Suppose A has a zero row.
Then $R_j \to -R_j \leftrightarrow A'$ Then $\det A = -\det A' = -\det A \implies \det A = 0$
(2) Case (i) If A is upper trianglular with one of the diagonal entries is 0, say a_{ii}.

$$\begin{bmatrix} a_{11}*** \\ 0a_{22}** \\ 000* \\ 000a_{44} \end{bmatrix} \to \begin{bmatrix} a_{11}*** \\ 0a_{22}** \\ 0000 \\ 000a_{44} \end{bmatrix}$$

Using row-operation using a_{44}, we now have a clear determinent 0 matrix by (1).
Case (ii) A upper triangular matric with all terms non-zero.

$$\begin{bmatrix} a**0b*00c \end{bmatrix} \xrightarrow[\text{Scale by } a^{-1}, b^{-1} \text{ and } c^{-1}]{} \begin{bmatrix} 1**01*001 \end{bmatrix} \xrightarrow[\text{Row recuctions}]{} \begin{bmatrix} 100010001 \end{bmatrix}$$

The last has a determinent 0 by defination. As we scaled by $\frac{1}{a}, \frac{1}{b}$ and $\frac{1}{c}$, we reverse the scaling to get the determinent abc. □

This is the fastest method of determinent finding in computer science and computer scientists get very happy looking at it.[11]

18.2 Another characterization of determinants

det can be thought of as a function of the rows of a matrix.

$$\det(\vec{v_1}, \vec{v_2}, \ldots, \vec{v_n}) = \det \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

18.3 Multi-linearity property

Let i be a natural number between 1 and n and fix $n-1$ vectors $\vec{v_1}, \vec{v_2}, \ldots, \vec{v_{i-1}}, \vec{v_{i+1}}, \ldots, \vec{v_n} \in \mathbb{R}^n$.
The the function $\mathbb{R}^n \to \mathbb{R}$

$$T_i(\vec{x}) = f \det(\vec{v_1}, \vec{v_2}, \ldots, \vec{v_{i-1}}, \vec{x}, \vec{v_{i+1}}, \ldots, \vec{v_n})$$

where $\vec{x}$ is weitten as a row on the RHS is linear map of the vector spaces and $f : M_{\mathbb{R}\times\mathbb{R}} \to \mathbb{R}$

18.4 Alternate defining properties

1. The determinant $\det(A)$ is multiplinear in the rows of A.

2. If A has two identical rows (or columns) then $\det(A) = 0$

3. The determinant of $T_{n\times n} = 1$,

Example 18.2. *Show that the equivalence of 'defining propertines' and the 'alternate defining properties'.*

Proof. TBD □

[11] According to prof. Aditya Karnataki. In reality, LU decomposition is beaten in practicality by Stressen's algorithm and in theory by the current bound of $O(n^{2.376})$ algorithm exists based on the CoppersmithWinograd algorithm.

Definition 18.3 (Multi-linear maps). *Let $V_1, \ldots, V_k, W$ be vector spaces.*
*A fucntion $f : V_1 \times V_2 \times \cdots \times V_k \to W$ is called **multilinear** if it is a linear in each of the argmenrs, when the emaining arguments are fixed.*

Example 18.4.
$$f(\alpha v_1 + v_1', v_2, \ldots, v_k) = \alpha f(v_1, v_2, \ldots, v_k) + f(v_1', v_2, \ldots, v_k)$$

Example 18.5.
$$\mathbb{R} \times \mathbb{R} \to \mathbb{R}$$
$$(a, b) \to ab$$
is multilinear with $k = 2$ or bilinear.

18.5 Co-Factor method of determinants

Let A be a $n \times n$ matrix.

Definition 18.6. *The i, j minor is denoted as A_{ij} is the $(n-1) \times (n-1)$ matrix obtained by deleting the i^{th} row and the j^{th} column.*

Definition 18.7. *The $(i, j)^{th}$ cofactor is $(-1)^{i+j} \det(A_{ij})$*

19 I was not able to take notes as my laptop is weak and can't run Zoom and Overleaf at the same time.

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20.1 Recall

Definition 20.1. *Charteristic polynomial of a linear operator $T : V \to V$ is*
$$p_T(x) = \det(xI - A)$$
$$\implies p_T(x) = x^n - (\text{tr}A)x^{n-1} + \cdots + (-1)^n \det A$$

Theorem 20.2.
$$\det(xI - A) = \det(xI - PAP^{-1})$$
where P is an invertible $n \times n$ matrix and all coefficients of chariteristic polynomial is independent of the basis.

Proof.
$$\det(xI - PAP^{-1}) = \det(P(xI - A)P^{-1}) = \det(P)\det(xI - A)\det(P^{-1}) = \det(xI - A)$$
□

Corollary: The eigenvalues of T are the roots of its characteristic polynomial similer matrices have the same eigen values.
Corollary: Diagonal entries of a trianguler matric are its eigenvalues.

Example 20.3. *Let K and W denote the kernel and image of an operator $V \to V$. Show that the following are equivalent.*
- $V = K \oplus W$
- $K \cap W = \{0\}$
- $K + W = V$

Corollary: If $\dim V = n$, then V has atmost n eiganvalues.

20.2 Vector spaces over a field F

Definition 20.4. *A field F is a set together with* $(+,0,-,\cdot,.,/,()^{-1})$.

We will take $\mathbb{F}=\mathbb{C}$.

Proposition 20.5. *If $F=\mathbb{C}$ and $V\neq 0$ then $T:V\to V$, then T has at least one eiganvalue and hejce at one eigan vector.*

Note, over a genral field, we may not have any eigan value.

Example 20.6. *R_θ or the rotation matrix by $\theta\in[0,2\pi)$.*
$p(x)=x^2-(2\cos\theta)x+1$, does not have real roots for $\theta\neq 0,\pi$.
But the operator $\mathbb{C}^2\to\mathbb{C}^2$ defined by $\mathbb{R}_\theta$ has complext eigen values $e^{i\theta}$ and $e^{-i\theta}$.

Proposition 20.7. *Every complex $n\times n$ matrix A is similar to an upper triangle matrices that is there exists an invertible matrix P (with complex entries) such that PAP^{-1} is upper triangular.*

Proof. A has ateast one eigen vector say v_1 with eigen value λ_1.
Extend $\{v_1\}$ to a basis of V.
In this basis,

$$A'=\begin{bmatrix}\lambda & * & \dots & * \\ 0 & d_{1,1} & \dots & d_{1,n-1} \\ \vdots & \vdots & \dots & \vdots \\ 0 & d_{n-1,1} & \dots & d_{n-1,n-1}\end{bmatrix}$$

By induction hypothesis on D which is clearly $(n-1)\times(n-1)$. This implies there exists a Q s.t. QDQ^{-1} is upper triangular.

$$P'=\left[10\dots 00q_1 1\dots q_1(n-1)\vdots\vdots\vdots\vdots\vdots 0q_n-1,1\dots q_(n-1),(n-1)\right]$$

P' makes A' upper triangular,
Similerly, P makes A upper trianguler. □

Remark 20.8. *If A is an $n\times n$ matrix with entries in a field F* s.t. *chearacteristic polunomidl of A is a product of linear factors (with entries in $\mathbb{R}$)* s.t.*PAP^{-1} is upper trianguler.*

Proposition 20.9. *An $n\times n$ matrix A is similer to a diagonal matrix if and only if there is a basis of $\mathbb{F}^n$ that consts of eigen vectos.*

Proof. $T(v_j)=\lambda_j v_j, j=1,2,\dots n$ then the matrix of T for the basis $(v_j)_{j=1}^n$ is

$$\begin{bmatrix}\lambda_1 0\dots 00 \\ 0\lambda_2\dots 00 \\ \vdots\vdots\vdots\vdots\vdots \\ 00\dots 0\lambda_n\end{bmatrix}$$

□

Moral: We can represent a linear operator by a diagonal matrix provided it has "enough" eiganvectors.

Theorem 20.10. *$T:V\to V$ over a field $\mathbb{F}$.*
If $P_T(x)$ has n disticit roots in $\mathbb{F}$, then there is a basis for V wr twhich matrix of T is a diagonal.

20.3 WHat about repeated eigan values?

Example 20.11.

$$A=\begin{bmatrix}41 \\ -21\end{bmatrix}$$

$$\det(xI-A)=\left[x-4-12x-1\right]$$

$$=(x-4)(x-1)+2=x^2-5x+6=(x-2)(x-3)$$

As roots are $2,3$, we get $\ker(A-2I)=\mathrm{Span}(()1,-2)^T$ *and* $\ker(A-3I)=\mathrm{Span}(()1,-2)^T$

Example 20.12. *Try to diagonalize*

$$\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$$

Both have repeared roots. But since B only has one eigenvector for the twice repeated eigan value 3, we can't have a basis or a diagonal form.

20.4 Geometric Multiplicity

$$\dim \ker(A - \lambda I) = \text{dimension of } \lambda \text{ eigenspace}$$

20.5 Algebric multiplicity

Number of times $x - \lambda$ appears as a factor of $\det(xI - A)$

Theorem 20.13. *Say $\{\lambda_i\}_{i=1}^r$ are eiganvalues for A. Take $\mathbb{F} = \mathbb{C}$. We have*

$$\sum_{i=1}^{r} g_i = n$$

where g_i is the geometric multiplicity of λ_i if and only if A is diagonalizable.

Theorem 20.14. *$a_i \geq g_i$ for each i where a_i is the algebric multiplicity of λ_i.*

Proof. Fix i.
Then $\lambda_i = \lambda$, $g_i = g$ and $a_i = a$ the take g many linearly independed eigenvectors associated with λ that is the basis vectors for $\ker(A - \lambda I)$.
Exted to a basis of V by chooseing some $v_{g+1}, \ldots, v_n$. □

Example 20.15. *See that weiting matrix of T in this baisis gives*

$$\det(xI - A) = (x - \lambda)^3 g(x)$$

where $g(x)$ is a polynomial.

Corollary: Since $\sum a_i = n$, from previous two theorems, A is a diagnoalizable if and only if $a_i = g_i$ for all i

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21.1 Duel Spaces

[12] **Recall:**[13] V, W vector spaces $\backslash\mathbb{F}$ then $\hom_{\mathbb{F}}(V, W)$ = Set of all linear maps / homomorphisms of V into W is a vector space over $\mathbb{F}$ with point-wise addition and scaler multiplication.

Lemma 21.1. $\dim_{\mathbb{F}} V = m, \dim_{\mathbb{F}} W = n \implies \dim_F \hom_{\mathbb{F}}(V, W) = mn$

Proof. Let $v_1, v_2, \ldots, v_m$ and $w_1, w_2, \ldots, w_n$ be bases for V and W respectivly.
For a vector $v \in V$, $v = \lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_m v_m$.
We define $T_{ij} : V \to W$ such that

$$T_{ij}(v_k) = \begin{cases} 0 & k \neq i \\ w_j & k = i \end{cases}$$

Claim: $T_i j$ constitute a basis for $\hom_{\mathbb{F}}(V, W)$

[12] Aw, we didn't start this one with recap...
[13] And we are back!

Proof. Let $S \in \hom_{\mathbb{F}}(V, W)$.

$$S(v_1) \in W \implies S(v_1) = \alpha_{11}w_1 + \alpha_{12}w_2 + \cdots + \alpha_{1n}w_n$$
$$S(v_2) \in W \implies S(v_2) = \alpha_{21}w_1 + \alpha_{22}w_2 + \cdots + \alpha_{2n}w_n$$
$$\vdots$$
$$S(v_i) \in W \implies S(v_i) = \alpha_{i1}w_1 + \alpha_{i2}w_2 + \cdots + \alpha_{in}w_n$$

Define

$$S_0 := \alpha_{11}w_1 + \alpha_{12}w_2 + \cdots + \alpha_{1n}w_n + \alpha_{21}w_1 + \alpha_{22}w_2 + \cdots + \alpha_{2n}w_n + \cdots + \cdots + \alpha_{m1}w_1 + \alpha_{m2}w_2 + \cdots + \alpha_{mn}w_n$$

$$\begin{aligned} \implies S_0(V_k) &= \alpha_{11}T_{11}(v_k) + \alpha_{12}T_{12}(v_k) + \cdots + \alpha_{mn}T_{mn}(v_k) \\ &= \alpha_{k1}w_1 + \cdots + \alpha_{kn}W_n \\ &= S(v_k) \forall 1 \le k \le m \end{aligned}$$

Thus, S_0 and S agree on basis V. Thus, it is spanning
Suppose, $\exists \beta_{ij}$ s.t.

$$\begin{aligned} \beta_{11}T_{11} + \beta_{12}T_{12} + \cdots + \beta_{mn}T_{mn} = 0 \\ \implies (\beta_{11}T_{11} + \beta_{12}T_{12} + \cdots + \beta_{mn}T_{mn})(v_i) = 0 \forall 1 \le k \le m \\ \beta_{k1}w_1 + \beta_{k2}w_2 + \cdots + \beta_{kn}w_n = 0 \end{aligned}$$

As $w_1, w_2, \ldots, w_n$ is a basis, $\beta_{k1} = \beta_{k2} = \cdots = \beta_{kn} = 0$.
Thus, it is spanning. □

□

Corollary: A basis for the space of $n \times m$ matrices is is given by matrices $\{M_{ij}\}$ with 1 in the $(i,j)^{th}$ entry and 0 elsewhere.
Cororollary: $\dim_{\mathbb{F}} \hom(V, \mathbb{F}) = m$ if V is finite dimensional, V and $\hom_{\mathbb{F}}(V, \mathbb{F})$ are isomorphic to each other becuse $V \cong \mathbb{F}^{\dim V} \cong \hom_{\mathbb{F}}(V, \mathbb{F})$

Remark 21.2. *The isomorphism has many shortcomings. There is no 'nice', 'universal' construction although it is still useful.* [14]

Definition 21.3. *If V is a vector space / $\mathbb{F}$ its dual space is then* $\hom_{\mathbb{F}}(V, \mathbb{F})$.

21.2 Notation for dual space

The notation is $\hat{V}$.
An element of $\hat{V}$ is called a linear functional on V. This is a function $f : V \to F$ and $v \to f(v)$ where v is a vector and $f(v)$ is a scaler.
V has a basis $v_1, \ldots, v_n$. Then define, $\hat{v_i} \in \hom(V, F)$ as

$$\hat{v_i}(v_j) = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

These are the $T'_{ij}s$ from the previous proof.
This implies $\hat{v_1}, \hat{v_2}, \ldots, \hat{v_k}$ is a basus for $\hat{V}$.
This is called **THE DUAL BASIS FOR** $v_1, v_2, \ldots, v_n$.
Corollary: If V is finite dimensional and $v \neq 0$ in V, then there is an element $f \in \hat{V}$ such that $f(v) \neq 0$.

Remark 21.4. *In fact, corollary is also true for V infinite dimension, but proof involves navigating logical minefields.*

[14] The nice and universal come from model theory. Why did sir mention it? Who knows?

21.3 Dual of a Dual

Let $v_0 \in V$. Let f vary over $\hat{V}$ then define $T_{v_0} \in \hom(\hat{V}, \mathbb{F})$ as

$$\hat{v} \to \mathbb{F}$$
$$f \to f(v_0)$$

Call $\hom_{\mathbb{F}}(\hat{v}, \mathbb{F}) = \hat{\hat{V}}$.
Then define $\psi : V \to \hat{\hat{V}} v \to T_v$.

$$T_{v+w}(f) = f(v+w) = f(v) + f(w) = T_v(f) + T_w(f)$$
$$\psi(v+w) = \psi(v) + \psi(w)$$

Similarly, $\psi(\lambda v) = \lambda \psi(v)$
This implies ψ is a linear map between V and $\hat{\hat{V}}$.

21.4 ψ isomorphism?

When is $\psi(v) = 0$?
Assume $T_v = 0$ for some V.
Then $T_v(f) = 0 \forall f \in \hat{V}$
$f(V) = 0 \forall f \in \hat{V}$.
For V finite dimension [15].
$\psi : V \xrightarrow{\sim} \hat{\hat{V}}$ is an isomorphism.
This is a canonical identification as no choices were made.

21.5 Inner Product Spaces

Let $\mathbb{F}$ be $\mathbb{R}$ or $\mathbb{C}$ now.
WHAT OTHER IDEAS ABOUT VECTORS OVER $\mathbb{R}$ AND $\mathbb{C}$?
Length
Perpendicularity
Both special cases of the notion of a dot product aka 'scaler product' aka 'inner product'.

Example 21.5. *$v = (x_1, x_2, x_3), w = (y_1, y_2, y_3) \in \mathbb{R}^3$, then*

$$v \cdot w = x_1 y_1 + x_2 y_2 + x_3 y_3$$
$$\text{length of } v = \sqrt{v \cdot v}$$
$$\cos\theta = \frac{v\dot{w}}{\sqrt{v\dot{v}}\sqrt{w\dot{w}}}$$

21.6 Formal properties over $\mathbb{R}$

1. $u\dot{v} = v\dot{u}$
2. $u\dot{u} = 0 \iff u = 0$
3. $u\dot{(}\alpha v + \beta w) = \alpha(u\dot{v}) + \beta(u\dot{w})$

21.7 Dot product over compelx

Notice, this defination doesn't work in complex.
For example, $V = (1, i, 0) \in \mathbb{C}^3$ has length 0. Also, it may have complex length by this method which doesn't make sense.
So we define dot product as:

$$v\dot{W} = x_1\overline{y_1} + x_2\overline{y_2} + x_3\overline{y_3}$$

This gives us:
1. $u\dot{v} = \overline{v\dot{u}}$
2. $u\dot{u} = 0 \iff u = 0$
3. $u\dot{(}\alpha v + \beta w) = \alpha(u\dot{v}) + \beta(u\dot{w})$

[15] and also infinite dimensional by the remark, but we don't talk about it.

21.8 In general field

Definition 21.6. $u, v \in V \exists < u, v > \in \mathbb{F}$ s.t.
1. $< u, v >= \overline{< v, u >}$
2. $< u, u > \geq 0$ *and* $< u, u >= 0 \iff u = 0$

3. $< \alpha u + \beta v, w >= \alpha < u, w > + \beta < v, w >$

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22.1 Recall

Definition 22.1. $u, v \in V \exists < u, v > \in \mathbb{F}$ s.t.
1. $< u, v >= \overline{< v, u >}$
2. $< u, u > \geq 0$ *and* $< u, u >= 0 \iff u = 0$

3. $< \alpha u + \beta v, w >= \alpha < u, w > + \beta < v, w >$

Remark 22.2. *Using 1 and 3,*

$$\begin{aligned} < u, \alpha v + \beta w > &= \overline{< \alpha v + \beta w, u >} \\ &= \overline{\alpha < v, u > + \beta < w, u >} \\ &= \overline{\alpha < v, u >} + \overline{\beta < w, u >} \\ &= \overline{\alpha} < v, u > + \overline{\beta} < w, u > \end{aligned}$$

Example 22.3. *Let* $V = \mathbb{F}^n, u = (\alpha_1, \alpha_2, \ldots, \alpha_n), v = (\beta_1, \beta_2, \ldots, \beta_n)$ *then,*

$$< u, v >= \alpha_1 \overline{\beta_1} + \alpha_2 \overline{\beta_2} + \cdots + \alpha_n \overline{\beta_n}$$

Example 22.4. *Let V be a set pf continuos real/complex valued functions on* $[0, 1]$.

$$f(t), g(t) \in V$$

$$\langle f(t) m g(t) \rangle = \int_0^1 f(t) \overline{g(t)} \mathrm{d}t$$

Check if this is indeed an inner product.

22.2 On Length and CSI

Definition 22.5. *length of* $v \in V$

$$\|v\| = \sqrt{\langle v, v \rangle}$$

Lemma 22.6.

$$\langle \alpha u + \beta v, \alpha u + \beta v \rangle =$$

Corollary

$$\|\alpha v\| = |\alpha| \|v\|$$

Lemma 22.7. *If* $a, b, c \in \mathbb{R}$ s.t. $a \geq 0$ *and* $a\lambda^2 + b\lambda + c \geq 0 \forall \lambda \in \mathbb{R}$ *then* $b^2 \leq ac$

Theorem 22.8 (Cauchy Scwarz Inequality). *If* $u, v \in V$, *then* $|\langle u, v \rangle| \leq \|x\| \cdot \|y\|$

Proof. (Case 1) If $u = 0$ then,
Claim: $\langle u, v \rangle = 0$ for any $v \in V$
In which case $|\langle u, v \rangle| = 0 = \|u\| \cdot \|v\|$
(Case 2) If $u \neq 0$, $\langle u, v \rangle \in \mathbb{R}$
Then for any $\lambda \in \mathbb{R}$

$$\begin{aligned} 0 \leq & \langle \lambda u + v, \lambda u + v \rangle \\ = & \lambda^2 \langle u, u \rangle + 2 \langle u, v \rangle + \langle v, v \rangle \end{aligned}$$

Let $a = \langle u,u\rangle, b = \langle u,v\rangle, c = \langle v,v\rangle$, by the quadratic idea, we have

$$\begin{aligned} b^2 &\le ac \\ \langle u,v\rangle^2 &\le \langle u,u\rangle\langle v,v\rangle \\ \implies |\langle u,v\rangle| &\le \|x\|\cdot\|y\| \end{aligned}$$

(Case 3) $u \neq 0$, $\langle u,v\rangle \notin \mathbb{R}$
Let $\alpha = \langle u,v\rangle$ then $\alpha \neq 0$ as $0 \in \mathbb{R}$.
Then $\langle \frac{u}{\alpha}, v\rangle = \frac{1}{\alpha}\alpha = 1$.
By the previos case, applied to $\frac{u}{\alpha}$ and v,

$$1 = |\langle \frac{u}{\alpha}, v\rangle| \le \frac{\|u\|}{|\alpha|}\|v\|$$

$$\begin{aligned} \therefore 1 &\le \frac{\|u\|}{|\alpha|}\|v\| \\ \therefore \alpha &\le \|u\|\|v\| \\ \implies |\langle u,v\rangle| &\le \|x\|\cdot\|y\| \end{aligned}$$

□

Corollary("Cauchy")

$$|\alpha_1\overline{\beta_1} + \alpha_2\overline{\beta_2} + \cdots + \alpha_n\overline{\beta_n}|^2 \le \left(|\alpha_1|^2 + |\alpha_2|^2 + \cdots + |\alpha_n|^2\right)\left(|\beta_1|^2 + |\beta_2|^2 + \cdots + |\beta_n|^2\right)$$

Corollary("Schwarz")

$$|\int_0^1 f(t)g(t)\mathrm{d}t|^2 \le \left(\int_0^1 |f(t)|^2\mathrm{d}t\right)\cdot\left(\int_0^1 |g(t)|^2\mathrm{d}t\right)$$

A philosophical idea is that average velocity over time has to be less than or equal to the average velocity over sidtance. Equality only occurs at a constant velocity.

22.3 Orthogonal

Definition 22.9. *If $u,v \in V$, then u is orthogonal to v of $\langle u,v\rangle = 0$.*
Similerly, v is orthogonal to u as $\langle u,v\rangle = \overline{\langle v,u\rangle}$

Definition 22.10. *If $W \subseteq V$ subspace, the orthogonal complement of W,*

$$W^\perp = \{x | x \in V, \langle x,w\rangle = 0 \forall w \in W\}$$

Lemma 22.11. *$W^\perp$ is a subspace of V*

Proof. $a,b \in W^\perp$, then for any $w \in W$,

$$\langle \alpha a + \beta b, w\rangle = \alpha\langle a,w\rangle + \beta\langle b,w\rangle = \alpha * 0 + \beta * 0 = 0$$

□

Lemma 22.12.

$$W \cap W^\perp = (0)$$

Proof. If $w \in W \cap W^\perp$, then $\langle w,w\rangle = 0$ which is a contradiction. □

Definition 22.13. *A set of vectors $\{v_i\} \in V$ is orthonormal set if*
1. $\langle v_i, v_i\rangle = 1$

2. $\langle v_i, v_j\rangle = 0$ *if* $i \neq j$

Lemma 22.14. *If $\{v_i\} \in V$ is an orthonormal set v_i are linearly independent. Further, if $w = \alpha_1 v_1 + \cdots + \alpha_n v_n$ then $\alpha_i = \langle w, v_i\rangle$.*

Proof. Suppose

$$\beta_1 v_1 + \beta_2 v_2 + \cdots + \beta_n v_n = \vec{0}$$

Take inner product with v_i

$$\beta_1\langle v_1, v_i\rangle + \cdots + \beta_i\langle v_i, v_i\rangle + \cdots + \beta_n\langle v_n, v_i\rangle = 0$$

The $LHS = \beta_i$ and $RHS = 0$ which implies $\beta_i = 0$ for all i.
Same computation gives the second part. □

Corollary: If $\{v_1, v_2, \ldots, v_n\}$ is an orthonormal set in V and if $w \in V$ then

$$u = W - \langle w, v_1\rangle v_1 - \langle w, v_2\rangle v_2 - \cdots - \langle w, v_n\rangle v_n$$

is orthogonal to each of $v_1, v_2, \ldots, v_n$.

Theorem 22.15 (Gram-Schmidt orthogonalisation process)**.** *Let V be a finite dimensional vector space,then V has an orthonormal set as a basis.*

Proof. Let $v_1, v_2, \ldots, v_n$ be any basis of V.
From this basis, we'll construct an orthonormal set of n vectors.
Define $w_1 = \frac{v_1}{\|v_1\|} \implies \langle w_1, w_1\rangle = 1$
Question For what value of α is $\alpha w_1 + v_1$ orthogonal to w_1

$$\langle \alpha w_1 + v_2, w_1\rangle = 0$$
$$\alpha\langle w_1, w_1\rangle + \langle v_2, w_1\rangle \alpha = -\langle v_2, w_1\rangle$$

Define $u_2 := v_2 - \langle v_2, w_1\rangle w_1$.

Also define $w_2 = \frac{u_2}{\|u_2\|}$ Similerly, $u_3 = v_3 - \langle v_3, w_1\rangle w_1 - \langle v_3, w_2\rangle w_2 \vdots\ u_{i+1} = v_{i+1} - \langle v_{i+1}, w_1\rangle w_1 - \langle v_{i+1}, w_2\rangle w_2 - \cdots - \langle v_{i+1} w_i\rangle w_i$ □

Remark 22.16. *Given r linearly independent vectors, we get an orthonormal ser with r vectors.*

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23.1 Recall

Theorem 23.1. *Let V be a finite dimensional inner product space, Then V has an orthonormal basis.*

Example 23.2. $V =$ ***set of polynomials in variable*** t ***with degree*** ≤ 2.
We define inner product as $\langle f(t), g(t)\rangle = \int_{-1}^{1} f(t)g(t)\mathrm{d}t$
A basis if $\{1, t, t^2\} \implies v_1 = 1, v_2 = t, v_3 = t^2$
Using Grahm-Schidt algorithm,

$$w_1 = \frac{v_1}{\|v_1\|} = \frac{1}{\sqrt{\int_{-1}^{1}\mathrm{d}t}} = \frac{1}{\sqrt{2}}$$

$$u_2 = v_2 - \langle v_2, w_1\rangle w_1 = t - \left(\int_{-1}^{1} t\frac{1}{\sqrt{2}}\right)\frac{1}{\sqrt{2}} = t$$

$$w_2 = \frac{u_2}{\|u_2\|} = \frac{t}{\sqrt{\int_{-1}^{1} t^2\mathrm{d}t}} = \sqrt{\frac{3}{2}}t$$

$$u_3 = v_3 - \langle v_3, w_1\rangle w_1 - \langle v_3, w_2\rangle w_2 = t^2 - \frac{1}{3}$$

$$w_3 = \frac{\sqrt{10}}{4}(3t^2 - 2)$$

23.2 Perpendicular Space

Theorem 23.3. *If V is a finite dimensional inner product space and if $W \subseteq V$ subspace, then $V = W + W^{\perp}$*

Proof. W itself is an inner prodict space.
This implies W has an orthonormal basis $w_1, w_2, \dots, w_r$.
For any $v \in V$,

$$v_0 = v - \langle v, w_1 \rangle w_1 - \langle v, w_2 \rangle w_2 - \cdots - \langle v, w_r \rangle w_r$$

is orthogonal to each w_i; hence all of W.
$\therefore v_0 \in W^{\perp}$ Then,

$$v = \left(\sum_{i=1}^{r} \langle v, w_i \rangle w_i \right) + v_0$$

Note $v \in V$ by hypothesis and the first part of the sum is from W and the second from $W^{\perp}$ □

Proof 2. Assume $\mathbb{F} = \mathbb{R}$ for simplicity. Let $v \in V$.
Claim(1): We can find $w_0 \in W$ s.t. $\|v - w_0\| \leq \|v - w\|$ for $w \in W$
Claim(2): If Such a w_0 exists, then $\langle v - w_0, w \rangle = 0 \forall w \in W$.

Proof of claim(2). Assume for $v \in V$, such w_0 exists so that $\|v - w_0\| \leq \|v - w\| \forall w \in W$.
Now, let $w \in W$. then $w + w_0 \in W$.

$$\begin{aligned}
\implies \langle v - w_0, v - w_0 \rangle \leq & \langle v - (w_0 + w), v - (w_0, w) \rangle \\
= & \langle w, w \rangle + \langle v - w_0, v - w_0 \rangle - 2\langle v - w_0, w \rangle \\
\implies 2\langle v - w_0, w \rangle \leq & \langle w, w \rangle
\end{aligned}$$

If m is any positive integer then $\frac{1}{m} w = \frac{w}{m} \in W$

$$\begin{aligned}
&\implies 2\langle v - w, \frac{w}{m} \rangle \leq \langle \frac{w}{m}, \frac{w}{m} \rangle \\
&\implies \frac{2}{m} \langle v - w, w \rangle \leq \frac{1}{m^2} \langle w, w \rangle \\
&\implies 2\langle v - w, w \rangle \leq \frac{1}{m} \langle w, w \rangle
\end{aligned}$$

As $m \to \infty$, $2\langle v - w, w \rangle \leq 0$.
As for all $w \in W$, $-w \in W$.

$$\begin{aligned}
&\implies 2\langle v - w_0, -w \rangle \leq 0 \\
&\implies -2\langle v - w_0, w \rangle \leq 0 \\
&\implies 2\langle v - w_0, w \rangle \geq 0
\end{aligned}$$

By squeeze theorem, $\langle v - w_0, w \rangle = 0$ for all $w \in W$. □

Proof of Claim (i). Let $u_1, u_2, \dots, u_k$ be a basis of W. Let $w \in W$ be written as $w = \lambda_1 u_1 + \lambda_2 u_2 + \cdots + \lambda_k u_k$.
Let $\beta_{ij} := \langle u_i, u_j \rangle$,
Let $\nu_i := \langle v_i, u_i \rangle$.
Thus,

$$\begin{aligned}
\langle v - w, v - w \rangle &= \langle v - \lambda_1 w_1 - \lambda_2 w_2 - \cdots - \lambda_k u_k, v - \lambda_1 w_1 - \lambda_2 w_2 - \cdots - \lambda_k u_k \rangle \\
&= \langle v, v \rangle + \sum_{i,j} \beta_{ij} \lambda_i \lambda_j - 2 \sum_i \nu_i \lambda_i
\end{aligned}$$

As w varies over all of W, λ_i's vary.
This makes it a function of λ_i's.
It is clearly non-negative and of degree 2.
Using calculus 2[16], we can say this has a minimum for some tuple.

$$(\lambda_{1,min}, \lambda_{2,min}, \dots, \lambda_{k,min})$$

Take w corresponding to this tuple. □

[16] According to sir, this is an ad for the second semester courses.

Alternate proof. Define a metric $S(x,y) := \|x-y\|$. V is a metric space.
Let $S := \{w \in W \mid \|v-w\| \le \|v\|\}$.
Claim(left as excercise) S is compact Then $f(v) = \|v-w\|$ is a continuous function.
$\therefore$ it achives a minimum over S call it W. □

□

23.3 Orthogonal Projection

Theorem 23.4. *Every vector $v \in V$ can be uniquly written as $v = w + u$ for $w \in W$ and $u \in W^{\perp}$*

Orthogonal projection from $V \to W$ is a map. We call it π.

$$\pi : V \to W, \pi(v) = w$$

Example 23.5. *Prove that π is linear map.*

π can also be defined at the unique linear map $\pi : V \to W$ such that

$$\pi(w) = w \text{ if } w \in W$$
$$\pi(u) = 0 \text{ if } u \in W^{\perp}$$

23.4 Projection Formula

If $w_1, w_2, \ldots, w_k$ is an orthonormal basisa for W,

$$\pi(v) = \langle w_1, v\rangle w_1 + \langle w_2, v\rangle w_2 + \cdots + \langle w_k, v\rangle w_k$$

Corollary: If V has orthonormal basis $v_1, v_2, \ldots, v_n$, then for $v \in V$

$$v = \langle v_1, v\rangle v_1 + \cdots + \langle v_k, v\rangle v_k$$

23.5 Bilinear form

For now, let V be a real vector space.
A **bilinear form on** V is a real valued function.

$$V \times V \to \mathbb{R}$$
$$(v,w) \to \langle v,w\rangle$$

linear in each variable.
$\langle rv_1 + v_2, w\rangle = r\langle v_1, w\rangle + \langle v_2, w\rangle$
$\langle v, sw_1 + w_2\rangle = s\langle v, w_1\rangle + \langle v, w_2\rangle$
Form of $\mathbb{R}^n$ given by

$$\langle X,Y\rangle = X^tAY, A \text{ is } n \times n \text{ matrix}$$

is an example of a binear form.
The dot product is the case $A = I$.
If a basis $v_1, \ldots, v_n$ is is given of V, then the matrix A corresponds to

$$A = (a)_{i,j}, a_{ij} = \langle v_i, v_j\rangle$$

Proposition 23.6. *If X and Y are coordinate vectors of v and w, then $\langle v,w\rangle = X^tAY$ where A is as before.*

Proof.

$$\begin{aligned}
\langle v,w\rangle &= \langle \sum_i v_i x_i, \sum_j v_j x_j\rangle \\
&= \sum_{i,j} x_i \langle v_i, v_j\rangle y_j \\
&= \sum_{i,j} x_i a_{ij} y_j \\
\implies \langle v,w\rangle &= X^tAY
\end{aligned}$$

□

Corollary: The inner product is symmetric if and only if A is symmetric.

24 14 November, 2024

24.1 Recall

$\langle v, w\rangle$ is a bilinear form $V \times V \to \mathbb{R}$.
A is a matrix associated to $\langle v, w\rangle$.

$$A = (a_{ij}), \quad a_{ij} = \langle v_i, v_j\rangle$$

[17]

24.2 Change of basis

[18]

24.3 Hermitian Forms

We are switching the defination of Inner product around here.

$$\langle cv_1, w_1\rangle = \bar{C}\langle v_1, w_1\rangle$$

$$\langle v_1 + v_2, w_1\rangle = \langle v_1, w_1\rangle + \langle v_2, w_1\rangle$$
$$\langle v_1, cw_1\rangle = c\langle v_1, w_1\rangle$$
$$\langle v_1, w_1 + w_2\rangle = \langle v_1, w_1\rangle + \langle v_1, w_2\rangle$$
$$\langle v_1, w_1\rangle = \overline{\langle w_1, v_1\rangle}$$
$$\langle v, v\rangle \in \mathbb{R} \forall v$$

Definition 24.1. *Adjoint $A*$ of a complex matrix (a_{ij}) defined by*

$$A* = (a*_{ij}) \iff a*_{ij} = \bar{a_{ji}}$$

$$\begin{bmatrix} 1 & 1+i \\ 2 & i \end{bmatrix} * = \begin{bmatrix} 1 & 2 \\ 1-i & -i \end{bmatrix}$$

Definition 24.2. *A square matrix is hermitian or* ***self-adjoint*** *if $A = A*$.
A real mi,ner os self adjoint if and on;y if it is symmetric.*

Definition 24.3. *The matrix of a Hermition form with respect to a basis $v_1, v_2 \ldots, v_n$*

$$A = (a_{ij}), \quad a_{ij} = \langle v_i, v_j\rangle$$

Proposition 24.4. *If X and Y are colmn vectrs of v and w, then*

$$\langle v, w\rangle = X^t AY$$

Definition 24.5. *A Hermitian form is* ***positive definite*** *if $\langle v, v\rangle > 0$ for every nonzero $v \in V$.
A Hermitian matrix is positive definite if $X * AX$ is positive for every non-zero complex colimn vector in $\mathbb{C}^n$.*

24.4 Change of Basis

$$\begin{aligned} \langle v, w\rangle &= X * AY \\ &= (PX') * A(PY') \\ &= (X') * (P * AP)Y' \end{aligned}$$

Corollory The standard Hermitian form $\bar{x_1}y_1 + \bar{x_2}y_2 + \cdots + \bar{x_n}y_n$ <fill here>

[17] CMI WIFI is a kludge. Please fucking work sometime.
[18] CMI WIFI is even a bigger kludge.

24.5 This may land you a PhD position.

Proposition 24.6. *The eiganvalues of a self-adjoint matrix are real numbers.*

Proof. Let X be an eiganvector with eiganvalue λ Then $X*AX = X*(\lambda X) = \lambda X*X$.
Note: $(\lambda X)* = \bar{\lambda}X*$.

$$\begin{aligned} A* &= A \\ X*AX = (X*A)X &= (X*A*)X = (AX)*X &= (\lambda X)*X = \bar{\lambda}X*X \text{ and } XX* \neq 0 \implies \lambda = \bar{\lambda} \end{aligned}$$

□

Corollary Trace and Determinent of a self adjoint matrix are real. **Corollary** The eiganvalues of a real symmetric matrix are real numbers.

24.6 Orthogonality

Definition 24.7. *v and w are orthogonal if and only if $\langle v,w\rangle = 0$.*

Example 24.8.

$$\langle X,Y\rangle = x_1y_1 + x_2y_2 + x_3y_3 - x_4y_4$$

$$\implies \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & -1 \end{bmatrix}$$

indefinite forms.
When a form is indefinite, a vector might be self-orthogonal.

Definition 24.9. $W \subseteq V$

$$W^{\perp} := \{v \in V \mid \langle v,w\rangle = 0 \forall w \in W\}$$

A null vector $v \in V$ is such that $\langle v,v'\rangle = 0 \forall v' \in V$.
The $V^{\perp}$ or null space of V is the space of all null vectors.
A form V is **non-degenerate** if $V^{\perp} = \{0\}$ if and only if for all non-zero v there is a vector v' such that $\langle v,v'\rangle \neq 0$.

Lemma 24.10. *A form on V is non-degenerate on $W \subseteq V \iff W \cap W^{\perp} = \{0\}$*

Proposition 24.11. *Let $\langle,\rangle$ be a non-degenerate Hermitian form on V. If $\langle v,w\rangle = \langle v',w\rangle \forall w \in V \implies v = v'$.*

Proposition 24.12. *(a) A vector v is a null vector if and only if its coordinate vector Y in the basis $(v_1, v_2, \ldots v_n)$ solves the equation*

$$AY = 0$$

where A is the matrix of $\langle,\rangle$ in the basis.
(b) The form is non-degenerate if and only if A is invertible.

Proof. If Y is such that $AY = 0$ then $X*AY = 0 \forall x$ that is Y is orthogonal to the entire space.
Conversely, if $AY \neq 0$, the it has at least one non-zero coordinate, say i^{th}.
Then $e*_i AY$ is that coordinate. □

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25.1 Recall

$\langle . \rangle$ symmetric over a real vector space or Hermitian on complex vector spae.
A matrix wrt some basis B
(a) a vector v is a null vevtor if and only if its coordinate vector Y wrt basis B solves $AY = 0$.
(b) The form is non-degenerate if and only if A is invertible.

Theorem 25.1. *(8.4.5 Artin) let* $\langle . \rangle$ *as before.*
$W \subseteq V$ *(a) The form is nondegenerate on W if and only if* $V = W \oplus W^{\perp}$.
(b) If the form is nondegerate on V and on W, then it is non-degenrate on $W^{\perp}$.

Lemma 25.2. *If* $\langle . \rangle$ *as befoe is not identcally zero, then* $\exists v \in V$ s.t. $\langle v, v \rangle \neq 0$.

Proof. $\exists x, y \in V$ s.t. $\langle x, y \rangle \neq 0$ and $\langle x, y \rangle = \langle y, x \rangle$ or $\overline{\langle y, x \rangle}$.
Then, $\langle x+y, x+y \rangle = \langle x, x \rangle + \langle x, y \rangle + \langle y, x \rangle + \langle y, y \rangle$ gives the answer. □

Theorem 25.3. *Let* $\langle . \rangle$ *as before. There exists an orthogonal basis for V where orthogonal basis* $:= \{v_1, \ldots, v_n\}$ s.t. $\langle v_i, v_j \rangle = 0$ *if* $i \neq j$

Proof. If $\langle . \rangle$ is identiacally zero, then every basis is orthogonal.
If $\langle < . > \rangle$ is not identically zero, then $\exists v \in V$ s.t. $\langle v, v \rangle \neq 0$.
Then choose $\{v\}$ as the first vector in the basis and let $W := \text{Span}(V)$.
On W the form is non-degenerte.

$$V = W \oplus W^{\perp}$$

By induction on dimension, $W^{\perp}$ has an orthogonal basis.

$$\{v_2, v_3, \ldots, v_n\}$$

Then append v_1 to it. □

Corollary: Let $\langle . \rangle$ be as before. Then there is an orthogonal basis $v_1, v_2, \ldots, v_n$ such that for each i, $\langle v_i, v_i \rangle = 1$ or -1 or 0.
$p :=$ Number of i such that $\langle v_i, v_i \rangle = 1$
$q :=$ Number of i such that $\langle v_i, v_i \rangle = -1$
$z :=$ Number of i such that $\langle v_i, v_i \rangle = 0$
(p, q) is the signature of the space and is invariant.
If the form is positive definite if and only if q and z are both 0.

25.2 Our first ever deep theorem

25.3 Spectral Theorem

Definition 25.4. *A real vector space with a positive definite symmetric bilinear form - "inner product space" is* ***Euclidian Space****. A complete inner product space is a* ***Hermitian space****.*

$T : V \to V$ a linear operator on a Hermitian space V.
A metric with respect to some basis B.
The adjoint operator $T* : V \to V$ is the operator whose matrix with respect to B is the adjoint matrix $A*$.

Definition 25.5. *A* ***normal matrix*** *is a complex matrix* A s.t. $A * A = AA*$

If $A* = A$, Hermitian matrix then A is normal.
$A* = A^{-1}$ 'Unitary matrix' then A is normal.

Lemma 25.6. *A* $n \times n$ *matrix* $\mathbb{C}$.
P $n \times n$ *unitary matrix.*
If A is normal, Hermitian or unitary, so is $P * AP$.

Definition 25.7. *A linear operator on a Hermitian space is* ***normal, Hermitian*** *or* ***unitary.***

If its matrix with respect to an orthonormal basis has the same property.
Proposition: (a) $\forall v, w \in V$,

$$\langle Tv, w \rangle = \langle v, T * w \rangle$$

$$< v, Tw >= \langle T * v, w \rangle$$

Proof. (a) Choose an orthonormal basis in which matrix of $\langle < . > \rangle$ is identity .
Let X and Y be column vectors of v and w in the same basis B, then

$$\langle Tv, w\rangle = (M_T X) * Y = X * M *_T Y$$

$$\langle v, T * w\rangle = X * M *_T Y$$

□

Proposition (b) T is normal $\iff \forall v, w \in V$

$$\langle Tv, Tw\rangle = \langle T * v, T * w\rangle$$

Proof. Substitute $T * v$ instead of v in the first equation of (a).
Substitute Tv instead of v in the second equation of (a).
Compare the above. □

Proposition (c) T is Hermitian $\iff \forall v, w \in V$

$$\langle Tv, w\rangle = \langle v, Tw\rangle$$

Proposition (d) T is unitary $\iff \forall v, w \in V$

$$\langle Tv, Tw\rangle = \langle v, w\rangle$$

26 Final Class

26.1 Recall

$T : V \to V$ linear operator.
V is Hermitian Space.
(a) $\forall v, w \in V, \langle Tv, w\rangle = \langle v, T * w\rangle$
$\langle T * v, w\rangle = \langle v, Tw\rangle$
(b) T normal $\iff \forall v, w \in V, \langle Tv, Tw\rangle = \langle T * v, T * w\rangle$; $TT* = T * T$
(c) T Hermitian $\iff \forall v, w \in V, \langle Tv, w\rangle = \langle v, Tw\rangle$; $T = T*$
(d) T is unitary $\iff \forall v, w \in V, \langle Tv, Tw\rangle = \langle v, w\rangle$; $TT* = I \iff T^{-1} = T*$

26.2 More recall

A subspace $W \subseteq V$ is ***T*-invariant** if $TW \subseteq W$.
From above, it follows that a normal/Hermitian/unitary operator will have the same property on an invarient subspace.

Proposition 26.1. *$T : V \to V$ a linear operator $V \to V$ and W T-invarient then $W^{\perp}$ is $T*$-invarient.*

Proof. **To show** if $u \in W^{\perp}$, then

$$T * u \in W^{\perp} \iff \langle w, T * u\rangle = 0 \forall w \in W$$

But $\langle w, T * u\rangle = \langle Tw, u\rangle$ where $Tw \in W$ and $u \in W^{\perp}$.
Thus, by definition, $\langle Tw, u\rangle = 0 \forall w \in W$. And we are done! □

Corollary If W is $T*$-invarient then $W^{\perp}$ is T-invarient.

Theorem 26.2. *Let T be a normal operator on a Hermitian Space V.*
Let v be an eigan-vector of T with eigan-value λ.
Then v is also an eigan-vector of $T*$ with eigan-value $\bar{\lambda}$.

Proof. (i) Say $\lambda = 0$
Then $Tv = 0$. **To show** $T * v = 0$.
$0 = \langle Tv, Tv \rangle = \langle T * v, T * v \rangle$
$\therefore T * v = 0$.
(ii) $\lambda \neq 0$.
Let $S = T - \lambda I$
Then v is an eigen vector of S with eigen value 0.
Also, $S* = T * -\bar{\lambda} I$.
Check S is a normal operator.
Then by case (i), the proof is finished.

□

Theorem 26.3 (Spectral Theorem for normal operators)**.** *Let T be a normal operator on the Hermitian space V.*
Then there is an orthonormal basis of V consisting of eiganvectors of T.

Proof. Choose an eigenvectors for v_1, for T.
Normalize its length to 1 by scaling if necessary.
Note v_1 is also eigenvector for $T*$.

$$w_1 = \text{Span}(v_1) \text{ is } T*\text{-invarient}$$
$$\therefore w_1^{\perp} \text{ is } T\text{-invarient}$$

T restricted to $w_1^{\perp}$ is also normal.
$\therefore w_1^{\perp}$ has an orthonormal basis consisting of eigencectors of T, say $\{v_2, v_3, \ldots, v_n\}$.
Adding v_1 to this set gives the required basis. □