Algebra II - End-Term - 100 Marks

April 29, 2025: Time: Three hours

1. **Write clearly. No marks will be give for unreadable statements.**
2. **Write all the necessary steps and clearly mention the results you are using.**
3. **Negative marks will be awarded for incomplete statements and unnecessary statements**
4. **Before beginning an answer to a question write the question number clearly like** $\boxed{\text{Question 1}}$

1. Let G be a group and let $H := \{g \in G | g^2 = 1\}$.

   (a) If G is abelian then show that H is a subgroup of G. (5 marks)

   (b) Does (1a) hold true if G is not abelian. (3 marks)

2. State and prove Sylow's third theorem. (8 marks) ✗

3. Let $F(X)$ be a free group on a set X. If X_1 and X_2 are two sets show that $F(X_1) \cong F(X_2)$ if and only if $|X_1| = |X_2|$. (6 marks)

4. Let $G = \mathbb{R}^*$. wrt multiplication

   addition

   (a) Show that every subgroup of G is either dense or discrete. (3 marks)

   (b) Show that the subgroup of G generated by 1 and $\sqrt{2}$ is dense. (3 marks)

   (c) Let H be a subgroup of G of angles. Show that H is either a cyclic subgroup of G or else it is dense in G. (3 marks)

5. Let G be a finite group and $|G| = p^e m$ where $e \geq 1$ and $p \nmid m$. Let H be a subgroup of G and $|H| = p^a$, $1 \leq a \leq e$. Show that $[N_G(H) : H] \cong [G : H]\ mod(p)$. (5 marks)

$|G| = |N_G(H)| + 0$

3
5

simple abelian thing

6. Let $n \geq 5$ and let A_n be the alternating group. Prove or disprove.

   (a) There exists a surjective homomorphism from A_5 onto $\mathbb{Z}/2Z$. (5 marks)

   (b) If $n > 6$ there exists a surjective homomorphism from A_n to $\mathbb{Z}/2Z$ (2 marks)

7. Let $p < q$ be primes and let G be a group of order $2pq$.

   (a) Prove that either a Sylow-p-subgroup H or a Sylow q-subgroup K is normal. (5 marks)

   (b) Prove that G has a normal subgroup of order pq. Is this subgroup cyclic subgroup? (6 marks)

   (c) What are all the homomorphisms from $\mathbb{Z}_2 \to \mathbb{Z}_{pq}$? (4 marks)

   (d) Suppose p does not divide $q-1$. Describe all groups of order $2pq$? Explain how you arrive at your answer. (7 marks)

   (e) Suppose $G = \mathbb{Z}_q \times D_p$ (D_p is the dihedral group with $2p$ elements). What are all the subgroups of G? You need to explain how you arrive at your answer. Which of them are normal? (8 marks)

   (f) Write down the class equation of $G \cong \mathbb{Z}_q \times D_p$. (6 marks)

8. Let G be a group and let H and K be normal subgroups of G. Consider

$$\frac{G}{H \cap K} \xrightarrow{\phi} \frac{G}{H} \times \frac{G}{K} \xrightarrow{\psi} \frac{G}{HK}$$

where $\phi(g(H \cap K)) = (gH, gK)$ and $\psi(aH, bK) = ab^{-1}(HK)$.

   (a) Show that ϕ is a homomorphism. (3 marks)

   (b) Describe $\ker(\phi)$. (2 marks) $H \cap K$

   (c) Show that ψ is a homomorphism? (3 marks)

   (d) Is ψ surjective? (2 marks)

   (e) Show that $Im(\phi) = ker(\psi)$ (6 marks)

   (f) State the Chinese Remainder Theorem. Applying (8a) to (8e) derive the Chinese Remainder Theorem. (6 marks)

$ab^{-1} = e$
$a = b$

$g(H \cap K)(H \cap K)$

$\phi(aH, bK)\phi(cH, dK)$
$= ab^{-1}(HK) \cdot cd^{-1}(HK)$