213
231 → {(1,2)3}
312 → {(1,2)(2,3)}
321 → {(2,3)(1,2)}
→ {(2,3)(1,2)(2,3)}

$\phi(k) = (n, n-1)(n-1, n-2) \cdots (k+1, k)$

$n-$

Algebra II - Mid-Term - 100 Marks

March 3, 2025: Time: Three hours

1. **Write clearly. No marks will be give for unreadable statements.**
2. **Write all the necessary steps and clearly mention the results you are using.**
3. **Negative marks will be awarded for incomplete statements and unnecessary statements**
4. **Before beginning an answer to a question write the question number clearly like $\boxed{\textbf{Question 1}}$**

1. Show that all the transpositions $(1,2),(2,3),\ldots,(n-1,n)$ generate S_n. (7 marks)

2. Let $A = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$ be elements in $GL_2(\mathbb{R})$. Determine the order of the following: (i) A, (ii) B, (iii) AB (18 marks)

3. Show that there is a bijection between HgK/K and $g^{-1}Hg/(g^{-1}Hg \cap K)$. (5 marks)

4. Let H be a subgroup of a G such that $|G/H| < \infty$. Show that there are only finitely many subgroups of the form gHg^{-1} where $g \in G$. (5 marks)

G-H acts on subgeus of G

5. Let $G = [e, \theta, a, b, c, \theta a, \theta b, \theta c\}$ where $a^2 = b^2 = c^2 = \theta$, $\theta^2 = e$, $ab = \theta ba = c$, $bc = \theta cb = a$, $ca = \theta ac = b$.

   $bc =$

   (a) Determine $Z(G)$, the center of G. (5 marks)

$\begin{matrix} 1 & -1 \\ 1 & 0 \end{matrix}$ $\begin{matrix} 1 & -1 \\ 1 & 0 \\ 0 & -1 \\ 1 & -1 \end{matrix}$

(b) Determine all subgroups of G. Which of them are normal? (8 marks)

(c) Prove or disprove: There exists an homomorphism from the Quaternion group to the symmetric group S_4. If yes explain how you get the homomorphism and compute the kernel. If no justify your answer. (10 marks)

6. Let H be a subgroup of G such that for all $g \in G$ there exists $g' \in G$ such that $Hg = g'H$. Then show that H is a normal subgroup of G. (5 marks)

7. Let G be a group of order $p^k m$ where p is a prime and $(p, m) = 1$. Let H be a subgroup of order p^k and K a subgroup of order p^d where $0 < d \leq k$ and $K \not\subset H$. Show that HK is not a subgroup of G. (5 marks)

8. Let p be a prime and let G be a group of order p^n. Suppose G acts on a finite set X. Let $Y = \{x \in X : gx = x \text{ for all } g \in G\}$. Then show that $|X| \cong |Y| (mod\ p)$. (5 marks)

9. Let G be a group and $\phi_g : G \to G$ be a map given by $\phi_g(x) = gxg^{-1}$.

(a) Show that ϕ_g induces an automorphism of G. (4 marks)

(b) Show that $G/Z(G) \cong Inn(G)$ where $Inn(G) := \{\phi_g : g \in G\}$. (4 marks)

(c) Let $G = \mathbb{Z}/6\mathbb{Z}$. Prove or disprove: $Aut(G) = Inn(G)$ (4 marks)

10. Let $p \geq 3$ be a prime and let G be a non-abelian group of order $2p^2$. Show that G is a non-abelian group which is a direct product of two sylow subgroups. Describe all the subgroups of this group(10 marks)

11. Let $G = \mathbb{Z}_p \times \mathbb{Z}_p$ and $G' = \mathbb{Z}_{p^2}$.

(a) Describe all homomorphisms from $G \to G'$. (3 marks)

(b) Which of them are isomorphisms? (2 marks)

$V_4 \to C_4$

$\phi(g) = e$

$\phi(a) = e$

$\phi(b) = \varepsilon$