Algebra II - Quiz 2 - 30 Marks

February 3, 2024: Time: One hour

1. Let G be a finite group. Show that the following are equivalent: (10 marks)

   (a) $|G|$ is prime.

   (b) $G \neq < 1_G >$ and G has no proper subgroups.

   (c) $G \cong \mathbb{Z}/p\mathbb{Z}$ for some prime p.

2. Let H and K be subgroups of G. Show that there is a bijection

$$\phi : H/(H \cap K) \to (HK)/K.$$

   $H \cap K$ is normal to H

   K is normal to HK

   (5 marks)

3. Let $G = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a, b \in \mathbb{Z}/p^2\mathbb{Z} \ \ \& \ \ a \cong 1 (mod\ p) \right\}$. Is G a group. If no justify. If yes determine the order of the group. (5 marks)

4. Let $n \geq 3$, $G = S_n$ be the symmetric group and A_n be the alternating group.

   (a) Let $\tau = (i_1 \ \ i_2 \ \ \cdots \ \ i_r)$ be a cycle and σ any permutation. Suppose $\sigma(i_r) = j_r$. Show that $\sigma\tau\sigma^{-1} = (j_1 \ \ \cdots j_r)$ (2 marks)

   (b) Prove or disprove: Let $G = A_4$. There exists proper subgroups H and K such that $A_4 = H \times K$. [Hint: Consider $\phi : H \times K \to G$](8 marks)

C_3 A_4/C_3

3412 4123

1234

$A_4 = C_3 \times V_4$

(123)

1234

2314 ⟵ 3124

$\langle (12)(34)$

$(13)(23) \rangle$