Algebra II - Quiz 3 - 25 Marks

April 7, 2025 : Time: One hour

1. Let N be a normal subgroup of G. If both N and G/N are finitely generated, then show that G is finitely generated. (5 marks)

2. Let G be a non-abelian group of order p^3.

   (a) What are all possible orders of $Z(G)$? Justify your answer(5 marks)

   (b) Let $x \notin Z(G)$. What is the order of $Z(x)$. Justify your answer. (5 marks)

3. Let $F(X)$ be a free group on a finite set X. Prove that each element of a free group has at the most finitely many roots; that is, for each $w \in F$, show that $\sqrt{w} := \{a \in F : a^n = w \text{ for some } n\}$ is finite. (5 marks)

4. Let p, q be prime numbers and let $2 < p < q$. Prove or disprove: Every group of order $2pq$ is simple. (5 marks)