Algebra II - Tutorial 1 - Date: 17/1/2025

1. Let x, y be elements of a group G. Assume that x, y, xy are elements of order 2. Show that $H = \{1, x, y, xy\}$ is a subgroup of order 4.

2. A permutation that can be expressed as a even product of 2-cycles is called an even permutation. Let A_n denote the set of all even permutations. Show that A_n is a subgroup of S_n.

3. List all subgroups of the dihedral group D_4. Which of them are normal. Is it possible to list all subgroups of D_n $(n \geq 3)$

4. List all subgroups of the symmetric group S_4. Which of them are normal?

5. Let $H = \{1, x^5\}$ be a subgroup of the dihedral group $G = D_{10} :=< x, y : x^{10} = y^2 = 1, yx = x^{-1}y >$. Compute all the left cosets G/H.

6. A n-th root of unity is a complex number z such that $z^n = 1$. (A cyclic group is a group generated by one element).

   (a) Show that the n-th roots of unity form a cyclic subgroup of $\mathbb{C}^\star$.

   (b) Determine the product of all n-th roots of unity.

7. Show that the elementary matrices (Page 71 of Artin)
   of first type, $E_1 = \big(\text{identity with } a \text{ at position } (j, i)\big)$ and
   third type, $E_3 = \big(\text{identity with } c \text{ at position } (i, i)\big)$ generate $GL_n(\mathbb{R})$.

8. Show that elementary matrices of first type generate $SL_n(\mathbb{R}) := \{A \in GL_n(\mathbb{R}) : det(A) = 1\}$.

9. (a) Let $G = S_n$. Use the row reduction method to show that transpositions generate the symmmetric group S_n.

   (b) Show that for $n \geq 3$, three cycles generate the alternating group A_n.

10. Prove that the set of all $n \times n$ matrices that have block form $M = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix}$ with $A \in GL_r(\mathbb{R})$ and $D \in GL_{n-r}(\mathbb{R})$ form a subgroup say H of $GL_n(\mathbb{R})$ and that the map that sends $\phi : H \to GL_r(\mathbb{R})$ that sends $M \in H$ to $A \in GL_r(R)$ is a homomorphism. What is its kernel? $(ker(\phi) := M \in H : \phi(H) = I_{r \times r})$.

Algebra II - Tutorial 1 - Date: 17/1/2025

1. If G is a group such that $(a \cdot b)^2 = a^2 \cdot b^2$ for all $a, b \in G$, show that G must be abelian.

2. If G is a group in which $(a \cdot b)^i = a^i \cdot b^i$ for three consecutive integers i for all $a, b \in G$, show that G is abelian.

3. Show that the conclusion of Problem 2 does not follow if we assume the relation $(a \cdot b)^i = a^i \cdot b^i$ for just two consecutive integers.

4. Let G be the set of real 2×2 matrices such that $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that $ad - bc \neq 0$ is a rational number. Show that G forms a group under this multiplication.

5. Let G be the set of real 2×2 matrices such that $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ such that $ad \neq 0$. Prove that G is a group. Is it abelian. Is it possible to construct a subgroup of order 4?

6. For all $n > 2$, construct a subgroup of order $2n$ in S_n. (Hint: Start with $n = 3$).

7. Let $\tau_{a,b} : \mathbb{R} \to \mathbb{R}$ be defined by $\tau_{a,b}(x) = ax + b$. Let $G = \{\tau_{a,b} : a \neq 0\}$. Show that G is a group under the composition of mappings. Find a formula for $\tau_{ab}\tau_{c,d}$.

8. Let $N = \{\tau_{1,b} \in G$ where G is as in Problem 7. Show that N is a subgroup of G. Show that for all $g \in G$ and $n \in N$, $ana^{-1} \in N$.

Algebra II - Tutorial 3 - Date: 27/1/2025

1. Let $G = S_4$ and G' the symmetric group on the three elements of S_4, $\Pi_1 = (12)(34)$, $\Pi_2 = (13)(24)$ and $\Pi_3 = (14)(23)$. Define $\phi :_4\rightarrow S_3$ as follows: $\phi(\sigma)$ is the permutation on $\{\Pi_1, \Pi_2, \Pi_{3\}}$ (Example 2.5.13) of Artin. Determine the kernel of ϕ What are all the subgroups which contain K.

2. Let $G =< x >$ be a cyclic group of order 12 and $G' =< y >$ a group of order 6. Let $\phi : G \rightarrow G'$ be given by $\phi(x^i) = y^i$. Exhibit the correspondance referred to in the correspondance theorem explicitly.

3. Let G be a group of order p^2. Show that it has atleast one subgroup of order p and if it has a unique subgroup of order p then it is a cyclic group.

4. Show that every subgroup of index 2 is a normal subgroup. Is a subgroup of index 3 a normal subgroup?

Algebra II - Tutorial 3 - Date: 27/1/2025

1. Let G be a group. An automorphism is an isomorphism $\phi : G \longrightarrow G$. Let $Aut(G)$: $\{\phi : G \longrightarrow G | \phi \text{ is an isomorphism}\}$.
   (a) Show that $Aut(G)$ is a group.
   (b) Determine $Aut(\mathbb{Z})$ where $n \geq 1$.
   (c) Show that $Aut(\mathbb{Z}_8) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$
   (d) Determine $Aut(D_4)$
   (e) Determine $Aut(Q_8)$ where Q_8 is the quartenion group.
2. Let H and K be subsets of G.
   (a) Prove that the double cosets partition G.
   (b) Do all double cosets have the same order
3. Let $G = GL_n(\mathbb{R})$, $L = \{\text{lower traingular matrices}\}$, $K = \{\text{upper traingular matrices}\}$. Show that G can be written as disjoint union of double cosets HPK where P is a permutation matrix
4. Decompose the set $\mathbb{C}^{2\times 2}$ of 2×2 complex matrices for the following operations of $GL_2(\mathbb{C})$.
   (a) By left multiplication
   (b) By Conjugation
5. Let $G = D_4$ be the diheral group of symmetries of a square.
   (a) What is the stabilizer of a vertex? What is the stabilizer of an edge?
   (b) G acts on the set of two elements consisting of diagonal lines. What is the stabilizer of a diagonal?

6. Describe the orbit and stabilizer of the matrix $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ under the conjugation action in $GL_n(\mathbb{R})$

Algebra II - Tutorial 5 - Date: 27/1/2025

1. Show that the group law on $H \rtimes_\phi K$ is associative.

2. Show that $S_n = A_n \rtimes \mathbb{Z}_2$

3. Show that $D_{2n} = D_n \times C_2$

4. $D_{2n} = C_n \rtimes C_2$ for a suitable homomorphism $\phi : C_2 \to Aut(C_n)$

5. Can we write $C_4 = C_2 \rtimes C_2$

6. Let x be a generator of C_4. Let $\phi : C_4 \to Aut(C_3)$ be the unique homomorphism such that $\phi_x(h) = h^{-1}$. Show that the semidirect product $C_4 \rtimes_\phi C_3$ is a nonabelian group of order 12 which is isomorphic to neither A_4 nor D_6. In fact this is the group $G =< x, y : x^4 = 1, y^3 = 1, xy = y^2x >$.

7. Let p and q be primes such that $p < q$.

   (a) Show that if p does not divide $q - 1$, then the only homomorphism from C_p to $Aut(C_q)$ is the trivial map (which sends every element to identity).

   (b) Show that if p divides $q - 1$, then there is a nontrivial homomorphism $\phi : C_p \to Aut(C_q)$. Further show that if $\psi : C_p \to Aut(C_q)$ is any other nontrivial homomorphism, then the semidirect products $C_q \rtimes_\phi C_p$ and $C_q \rtimes_\psi C_p$ are isomorphic. Deduce that, if p divides $q-1$, then up to isomorphism there is only one nonabelian group of order pq

8. Show that the quartenion group Q_8 a not a semidirect product of any of its two proper subgroups.

9. Determine whether $C_6 = C_3 \rtimes C_2$

10. Determine whether $S_3 = C_3 \rtimes C_2$

11. Show that $GL_n(\mathbb{R}) = SL_n(\mathbb{R}) \rtimes diag(a, 1, \ldots, 1)$ where $a \in \mathbb{R} - \{0\}$ and .

Algebra II - Tutorial 5 - Date: 27/1/2025

1. Let G be a group and let $Inn(G)$ denote the set of all inner automorphisms of G. SHow that $Inn(G)$ is a normal subgrouo of $Aut(G)$.

2. Suppose $G/Z(G)$ is cyclic. Show that R is abelian.

3. Let G be a group of order pn where p is prime and $p > n$. Let H be a subgroup of G. Then H is normal in G.

4. Let G be a nonabelian group of order p^3.

   (a) What are the possibilities for $Z(G)$, the center of G.

   (b) Let $x \in G \setminus Z(G)$. What is the order of $Z(x)$.

   (c) What are the possibilities of the class equation of G?

   (d) What is the class equation of (i) D_4 (ii) Q_8?

   Let G be a group of order 12 and K a Sylow-3 subgroup. Consider $G/K = \{g_1K = K, g_2K, g_3K, g_4K\}$ and let $Sym(G/K)$ denote the symmetric group on the set G/K. Let $\phi : G \to Sym(G/K)$ be the map given by $\phi(g)(g_iK) := gg_iK$.

   (a) Show that ϕ is a homomorphism

   (b) Show that ϕ is injective if and only if K is not a normal subgroup of G.

   (c) If ϕ is injective, show that $G \cong A_4$.

5. Let $p \neq q$ be prime numbers and let G be a group of order p^2q. Show that G has a normal sylow subgroup.

Algebra II - Tutorial 8 - Date: 24/3/2025

1. Prove that a group G is free on a subset X if and only if X generates G and no reduced word in $X \cup X^{-1}$ of positive length is the identity. ($\ell(w) :=$ length of the word is the length of a reduced word)

   Hint:—Show that the inclusion map $X \to G$ extends to a unique isomorphism $F(X) \to G$

2. Show that free groups F(X) do not have elements of finite order $>$. In fact, if $a^n = b^n$ for a, b in a free group F, prove that $a = b$.

   Hint: Need to consider words of the form $w = w_1^{-1}w_2w_1$ and consider w^n

3. Let F be a free group let $w \in F$ Show that $\{a \in F : a^n = w\}$ is finite

   Hint: Consider $\ell(w)$ and $\ell(a^n)$

4. Show that in a free group, two commuting elements a, b must satisfy $a = c^u$ and $b = c^v$ for some element c and some integers u, v. In particular, a free group has nontrivial center if and only if its rank is 1.

5. If a, b are elements in a free group F, and satisfy $a^pb^q = b^qa^p$ for some non-zero integers p, q then prove a, b are integer powers of a common element.

6. If $w \neq 1$ in a free group F, then prove that the centralizer $C(w)$ is an infinite cyclic group.

7. If $w \neq$ in a free group, show that w cannot be conjugate to w^{-1}.

8. If N is a normal subgroup of a group G such that G/N is free, then prove that there exists a subgroup H of G satisfying $G = HN$ and $H \cap N = \{1\}$.

9. Let H be a subgroup of infinite index in a free group F. Show that for each subgroup $K \neq \{1\}$ of F, $H \cap K \neq \{1\}$.