Quiz 1- Monday 8/9/2025,

1. Let p be a prime and let $R = \mathbb{Z}[x]$ and $S = \mathbb{Z}[x, 1/p]$. Show that S is an R-module. (3 marks)

2. Let R and S be commutative rings with identity. Let $U(R)$ (resp. $U(S)$) denote the set of units in R (resp. S).

   (a) Show that $U(R \times S) \cong U(R) \times U(S)$. (3 marks)

   (b) Let R be a PID and let I be any non-zero proper ideal in R. Show that there exists maximal ideals $\mathfrak{m}_1, \ldots, \mathfrak{m}_s$ and positive integers $n_1, \ldots, n_s$ such that $U(R/I)$ and $U(R/\mathfrak{m}_1^{n_1}) \times \cdots \times U(R/\mathfrak{m}_s^{n_s})$ are isomorphic. (5 marks)

   (c) Let n be any integer. Use (2a) and (2b) to determine the number of units in $\mathbb{Z}/n\mathbb{Z}$. (3 marks)

3. Consider the ring $\mathbb{C}[x;\sigma]$ where σ denotes complex conjugation on $\mathbb{C}$ and for any $a \in \mathbb{C}$, $xa = \sigma(a)x$. Prove or disprove: $A/A \cdot (x^4 + 1)$ is a division ring. You need to explain how you arrive at your answer. (5 marks)

4. For any ring R, let $R[[x]]$ denote the power series ring. Let $\mathbb{R}$ denote the real numbers.

   (a) Let P be a prime ideal in $\mathbb{R}[x]$. Prove or disprove $P\mathbb{R}[[x]]$ is a prime ideal in $\mathbb{R}[[x]]$. (3 marks)

   (b) Let $\mathfrak{m}$ is a maximal ideal in $\mathbb{R}[[x]]$. Prove or disprove: $\mathfrak{m} \cap \mathbb{R}[x]$ is a maximal ideal in $\mathbb{R}[x]$. (3 marks)

$(a+bi)x = (a-bi)x$

$P \in \mathbb{R}$

$(x+b)(cx+d)\, a \cdot b = P\mathbb{R}[[x]]$

$\langle 1 \rangle = I$

$\frac{(x^4+1)}{\mathbb{C}[x]}$

$(a,b) \times (c,d) = (1,k)$ $\quad \frac{1}{x^4+1}$ $\quad x^4+1$

$ac x^2 + (ad+bc)x$

$a \in P$

$ab \in P$

$\Rightarrow a \in P$ or $b \in P$

$ad = -bc$

$\Leftrightarrow (ac, bd) = (1,1)$

$\Leftrightarrow ac = 1 \quad bd = 1$

$P\mathbb{R}[[x]]$

$(x-i)(x+i)$

$\Rightarrow c = a^{-1} \quad b = d^{-1}$

$\overline{\mathbb{C}}[x]\ \mathbb{R}[[x]] = \overline{\mathbb{C}}[P(x)]$

$ab \in$

$(x^2-i)(x^2+i)$

$U(R/m_1^{n_1}) \times \cdots$

$(x-i)(x+i) \quad (x-\sqrt{i})(x+\sqrt{-i})$

$12 = 2^2 \cdot 3$

1 7
5 11

$R \to R/I$

$U(2^2) \cdot U(3)$ PID $\subseteq$ UFD

$\downarrow 2 \quad \downarrow 2$

$4 \to 3 \quad 0(12)$

$(p-1)$

$p^{e_1}\left(1-\frac{1}{p}\right)$

$p^{e_1} - p^{e_1 - 1}$

$\frac{\mathbb{C}[x]}{\mathbb{C}[x](x^4+1)}$

$\frac{(x^4+1)+a}{(x^4+1)+b} = (x^4+1)+c$

$\frac{(x^6-1)}{(x+1)(x^5-1)}$

$\frac{x^4+x^2+1}{x^4+x^3+x^2+1}$

$\frac{x^6-1}{x-1}$

$\frac{x^5-1}{x-1}$

$x+1$

$(x^4+1)+a = (x^4+1)+bc$