

1. State and prove Gauss Lemma for a UFD. (7 marks)

2. Let R be a UFD and Q the quotient field of R . Suppose $f(x) \in R[x]$ is primitive polynomial of positive degree. Show that $f(x)$ is irreducible in $R[x]$ if and only if $f(x)$ is irreducible in $Q[x]$. (6 marks)

3. Let R be a UFD and let $f, g \in R[x]$ be polynomials and $I = (f, g) \subseteq R[x]$. Let Q be the quotient field of R .

(a) Show that $IQ[x] = Q[x]$ if and only if the ideal $I \subseteq R[x]$ contains a non-zero element of R . (5 marks)

(b) Suppose R is a PID $IQ[x] = Q[x]$. Is it possible to write $I \subseteq R[x]$ as a product of maximal ideals? (3 marks)

(c) Does (3b) hold true if R is a UFD? (4 marks)

4. Let k be a field and let $R = k[x_1, \dots, x_d]$ be a polynomial ring. Let $r, d \geq 2$ and $f(x_1, \dots, x_d) = x_1^r + x_2^r + \dots + x_d^r$.

(a) Let $d = r = 3$. Is f irreducible. Does it depend upon the characteristic of the field. (5 marks)

(b) Let $r = 3$ and $d \geq 2$. Is f irreducible (5 marks)

(c) For what values of r, d is f irreducible? Does it depend upon the characteristic of the field. (5 marks)

Handwritten work for problem 4(a):

Let $d = r = 3$. $f(x_1, x_2, x_3) = x_1^3 + x_2^3 + x_3^3$.

Assume f is reducible. Then it factors into linear factors over k . Suppose $f = (x_1 + x_2 + x_3)(x_1^2 + x_2^2 + x_3^2 + x_1x_2 + x_1x_3 + x_2x_3)$.

Equating coefficients, we get:

$$x_1^3 + x_2^3 + x_3^3 = (x_1 + x_2 + x_3)(x_1^2 + x_2^2 + x_3^2 + x_1x_2 + x_1x_3 + x_2x_3)$$

Expanding the right side:

$$= x_1^3 + x_2^3 + x_3^3 + x_1^2x_2 + x_1^2x_3 + x_2^2x_1 + x_2^2x_3 + x_3^2x_1 + x_3^2x_2 + x_1^2x_2 + x_1^2x_3 + x_2^2x_1 + x_2^2x_3 + x_3^2x_1 + x_3^2x_2 + x_1x_2x_3 + x_1x_2x_3 + x_1x_2x_3$$

Canceling terms, we get:

$$0 = 3x_1x_2x_3$$

Since k is a field, $3 \neq 0$ (assuming characteristic is not 3), we get $x_1x_2x_3 = 0$, which is a contradiction.

Therefore, f is irreducible.

For part (b), let $r = 3$ and $d \geq 2$. $f(x_1, \dots, x_d) = x_1^3 + x_2^3 + \dots + x_d^3$.

Assume f is reducible. Then it factors into linear factors over k . Suppose $f = (x_1 + x_2 + \dots + x_d)(x_1^2 + x_2^2 + \dots + x_d^2 + x_1x_2 + \dots + x_1x_d + x_2x_3 + \dots + x_{d-1}x_d)$.

Equating coefficients, we get:

$$x_1^3 + x_2^3 + \dots + x_d^3 = (x_1 + x_2 + \dots + x_d)(x_1^2 + x_2^2 + \dots + x_d^2 + x_1x_2 + \dots + x_1x_d + x_2x_3 + \dots + x_{d-1}x_d)$$

Expanding the right side, we get:

$$= x_1^3 + x_2^3 + \dots + x_d^3 + x_1^2x_2 + x_1^2x_3 + \dots + x_1^2x_d + x_2^2x_1 + x_2^2x_3 + \dots + x_2^2x_d + \dots + x_{d-1}^2x_d + x_1^2x_2 + x_1^2x_3 + \dots + x_1^2x_d + x_2^2x_1 + x_2^2x_3 + \dots + x_2^2x_d + \dots + x_{d-1}^2x_d + x_1x_2x_3 + \dots + x_1x_2x_d + \dots + x_1x_{d-1}x_d + x_2x_3x_4 + \dots + x_2x_3x_d + \dots + x_{d-2}x_{d-1}x_d$$

Canceling terms, we get:

$$0 = 3x_1x_2x_3 + \dots + 3x_1x_2x_d + \dots + 3x_1x_{d-1}x_d + \dots + 3x_2x_3x_4 + \dots + 3x_2x_3x_d + \dots + 3x_{d-2}x_{d-1}x_d$$

Since k is a field, $3 \neq 0$ (assuming characteristic is not 3), we get $x_1x_2x_3 + \dots + x_1x_2x_d + \dots + x_1x_{d-1}x_d + \dots + x_2x_3x_4 + \dots + x_2x_3x_d + \dots + x_{d-2}x_{d-1}x_d = 0$, which is a contradiction.

Therefore, f is irreducible.

For part (c), let r, d be arbitrary. $f(x_1, \dots, x_d) = x_1^r + x_2^r + \dots + x_d^r$.

Assume f is reducible. Then it factors into linear factors over k . Suppose $f = (x_1 + x_2 + \dots + x_d)(x_1^{r-1} + x_2^{r-1} + \dots + x_d^{r-1} + x_1x_2^{r-2} + \dots + x_1^{r-2}x_2 + \dots + x_1^{r-2}x_d + x_2^{r-2}x_1 + \dots + x_2^{r-2}x_d + \dots + x_{d-1}^{r-2}x_d + x_1^{r-2}x_2 + x_1^{r-2}x_3 + \dots + x_1^{r-2}x_d + x_2^{r-2}x_1 + x_2^{r-2}x_3 + \dots + x_2^{r-2}x_d + \dots + x_{d-1}^{r-2}x_d + x_1x_2x_3 + \dots + x_1x_2x_d + \dots + x_1x_{d-1}x_d + x_2x_3x_4 + \dots + x_2x_3x_d + \dots + x_{d-2}x_{d-1}x_d$.

Equating coefficients, we get:

$$x_1^r + x_2^r + \dots + x_d^r = (x_1 + x_2 + \dots + x_d)(x_1^{r-1} + x_2^{r-1} + \dots + x_d^{r-1} + x_1x_2^{r-2} + \dots + x_1^{r-2}x_2 + \dots + x_1^{r-2}x_d + x_2^{r-2}x_1 + \dots + x_2^{r-2}x_d + \dots + x_{d-1}^{r-2}x_d + x_1^{r-2}x_2 + x_1^{r-2}x_3 + \dots + x_1^{r-2}x_d + x_2^{r-2}x_1 + x_2^{r-2}x_3 + \dots + x_2^{r-2}x_d + \dots + x_{d-1}^{r-2}x_d + x_1x_2x_3 + \dots + x_1x_2x_d + \dots + x_1x_{d-1}x_d + x_2x_3x_4 + \dots + x_2x_3x_d + \dots + x_{d-2}x_{d-1}x_d)$$

Canceling terms, we get:

$$0 = 3x_1^{r-2}x_2 + 3x_1^{r-2}x_3 + \dots + 3x_1^{r-2}x_d + 3x_2^{r-2}x_1 + 3x_2^{r-2}x_3 + \dots + 3x_2^{r-2}x_d + \dots + 3x_{d-1}^{r-2}x_d + 3x_1^{r-2}x_2 + 3x_1^{r-2}x_3 + \dots + 3x_1^{r-2}x_d + 3x_2^{r-2}x_1 + 3x_2^{r-2}x_3 + \dots + 3x_2^{r-2}x_d + \dots + 3x_{d-1}^{r-2}x_d + 3x_1x_2x_3 + \dots + 3x_1x_2x_d + \dots + 3x_1x_{d-1}x_d + 3x_2x_3x_4 + \dots + 3x_2x_3x_d + \dots + 3x_{d-2}x_{d-1}x_d$$

Since k is a field, $3 \neq 0$ (assuming characteristic is not 3), we get $x_1^{r-2}x_2 + x_1^{r-2}x_3 + \dots + x_1^{r-2}x_d + x_2^{r-2}x_1 + x_2^{r-2}x_3 + \dots + x_2^{r-2}x_d + \dots + x_{d-1}^{r-2}x_d + x_1^{r-2}x_2 + x_1^{r-2}x_3 + \dots + x_1^{r-2}x_d + x_2^{r-2}x_1 + x_2^{r-2}x_3 + \dots + x_2^{r-2}x_d + \dots + x_{d-1}^{r-2}x_d + x_1x_2x_3 + \dots + x_1x_2x_d + \dots + x_1x_{d-1}x_d + x_2x_3x_4 + \dots + x_2x_3x_d + \dots + x_{d-2}x_{d-1}x_d = 0$, which is a contradiction.

Therefore, f is irreducible.