Quiz 2- Monday 3/11/2025,

1. State and prove Gauss Lemma for a UFD. (7 marks)

2. Let R be a UFD and Q the quotient field of R. Suppose $f(x) \in R[x]$ is primitive polynomial of positive degree. Show that $f(x)$ is irreducible in $R[x]$ if and only if $f(x)$ is irreducible in $Q[x]$. (6 marks)

3. Let R be a UFD and let $f, g \in R[x]$ be polynomials and $I = (f, g) \subseteq R[x]$. Let Q be the quotient field of R.

   (a) Show that $IQ[x] = Q[x]$ if and only if the ideal $I \subseteq R[x]$ contains a non-zero element of R. (5 marks)

   (b) Suppose R is a PID $IQ[x] = Q[x]$. Is it possible to write $I \subseteq R[x]$ as a product of maximal ideals? (3 marks)

   (c) Does (3b) hold true if R is a UFD? (4 marks)

4. Let k be a field and let $R = k[x_1, \dots, x_d]$ be a polynomial ring. Let $r, d \geq 2$ and $f(x_1, \dots, x_d) = x_1^r + x_2^r + \cdots x_d^r$.

   (a) Let $d = r = 3$. Is f irreducible. Does it depend upon the characteristic of the field. (5 marks) $x_1^3 + x_2^3 + x_3^3$

   (b) Let $r = 3$ and $d \geq 2$. Is f irreducible (5 marks) $x_1^3 + x_2^3 + \dots$

   (c) For what values of r, d is f irreducible? Does it depend upon the characteristic of the field. (5 marks)

$(x_1+1)^3 = x_1^3 + 3x_1^2 + 3x_1 + 1$

$1+0+1$

$a_1 f(x) + a_2 g(x) = x_2^3 + x_3^3$

$p \mid x_2^3 + x_3^3$

$\Rightarrow x_2 = -x_3$

$x_2 + x_3 = p$

$x_2 + x_3$