Grading schemes for final exam

Q1 7.5 points

(a) 1.5 True
$\limsup x_i$ = limit of the decreasing sequence s_1, s_2 = inf of all s_i
Each s_i and their inf are real numbers because the sequence is bounded.
We also know that $\limsup x_i$ is the supremum of S = set of all sub sequential limits of x_i
S is closed so it contains its sup, which means $\limsup x_i = \inf s_i$ is a sub sequential list of given sequence.

0.5 limsup using tails, is finite
0.5 limsup as sup of subseq limits
0.5 set of subseq limits closed, so contains sup

(b) 1.5 False
All points for counterexample + showing four properties: f and g differentiable, both go to 0 at 0, f'/g' has no limit at 0 but f/g does.

(c) 1.5 True

S is bounded and closed so compact, so sequentially compact
0.5 S is closed
0.5 and bounded so compact
0.5 so sequentially compact

(d) 1.5

0.5 Taylor linear polynomail
0.5 correctly explained quadratic error term
0.5 No because not differentiable at $a = 0$

(e) 1.5

0.5 clear statement of Lebesuge criterion: integrable iff discontinuities form a zero set
0.5 fg is cont wherever f and g ar both cont, discontinuities are at most union of those of f and g, union of two zero sets is a zero set
0.5 $|f|$ is continuous wherever f (because absolute value is continuous), so discontinuities can only become fewer

Q2 6 points

(a) 2 = 1 + 1
Yes by intermediate value theorem (if nonconstant)
Yes + No. By extreme value theorem f(x) = [min value, max value]

(b) 2 = 1.5 + 0.5
Yes: any interval is connected, so image connected, so must be an interval (or a single point if f is constant)
No e.g. image of (0,1] under 1/x

(c) 2 = 0.5 each
f(X) = (-c,c) possible for X = R by arctan type of function

f(x) = (a,b] and [a,b) both possible for x = (-infty,0] , resp [0,infty) again by arctan
or by e^(1/x^2)
f(x) = [a,b] possible for X = R by ensuring f is bounded with a max and a min e.g. f = sin

Q3 6 points This is Rudin theorem 3.55

1 to fill Cauchy condition for absolute convergence of summation la_il

3 for carefully explaining how the cancellation occurs (of terms 1 to N and maybe others, so that the remaining part is just a bunch of a_i for i beyond N, some with + and some with - sign. Then use triangle inequality and choice of N

2 for showing how the t_k have the same limit as s_k explicitly, again with triangle inequality (sketch ok as long as understanding is clear)

Q4 total 7 points. Standard "divided by 3" argument, but has to be used carefully.

Given e > 0, need to find d > 0 such that lx-yl < d ensures lg(x)- g(y)l < e
Use unif continuity for Q get d such that la-bl < 3d ensures lf(a)- f(b)l < e
Claim: this d works
Take x_n, y_n in Q approaching x and y respectively. we have lim f(x_n) = g(x) and lim f(y_n) = g(y)
Find large enough n meeting four conditions
in the domain x_n is within d of x and y_n is within d of y
in the range f(x_n) is within e/3 of g(x) and f(y_n) is within e/3 of g(y) (so f(x)
If x and y are < d apart, then x_n and y_n are < 3d apart by triangle inequality in the domain
So lf(x_n)- f(y_n)l < e/3. Now range inequality in the range gives lf(x) - f(y)l < e.

I'd say each of the following deserves ~1 point, but of course total credit depends on coherence.
Maybe 3/7 can be cobbled together with bits and pieces, but I'd look for some overall understanding to give more than 3.5/7

Some awareness of using division by 3 with triangle inequality
Awareness that it needs to be used both in the domain and the range
Clear use of definition of extension g using sequences

Clear use of uniform continuity over Q with the previous step
Explicit mention of large enough n to get x_n close to x and f(x_n) to g(x)

Q5 Total 7.5

5a = 2
5b = 2
a and b marks only for correct reasoning, partial as appropriate, cut marks (0.5?) only once

in a and b if they never make the partial sums explicit in applying 7.12 and 7.17. Consider not cutting if they make it explicit only in one part but not the other.

5c = 3.5

 2 for integrability over [0,pi/2] with justification and value
 1.5 for rest, because F(b) is immediate from earlier work
 (so they just have to see two sequences of b giving different limits)

Q6 Total 6

6a = 2.5, can give 0.5 just for stating what needs to be proved
6b = 3.5
0.5 for some indication of example being on correct track (even without justification, partially right ok)
1.5 for differentiable and derivative unbounded (not if the example itself is on wrong track)
1.5 for uniform continuity

Q7 total 6 points

4 for proving from basic principles that one gets the correct answer for continuous function e.g.

 1 for given epsilon, use uniform continuity to get N for which x-y < 1/N ensures f(x) - f(y) < epsilon
 1 For n > N, in each subinterval sup M - inf m is at most epsilon (even strictly less as m and M are attained by extreme value theorem)
 1 so (upper sum U) - (lower sum L) is at most epsilon times length of the interval, so here U - L is at most epsilon
 1 now left hand sum and the integral are both sandwiched between U and L, so differ by at most epsilon, which proves the statement

2 for a counterexample

 1 to show school internal exists
 1 to show not integrable (cut 0.5 for a correct function without proof)

Q8 6 points

1.5 (a) Rudin 3.54. n-th root of 1/n has lim (and so limsup) = 1 (by 3.20c, but they can state without justification), so R = 1/1 = 1.

 Procedure should be made clear even if the theorem is not explicitly mentioned.
 Can also use ratio test, but then need to explain using 3.34, not just directly take 1/limit of ratio. (I'd cut 0.5 if they are cavalier here.)

4.5 (b,c) depending on answers and reasoning.

 1 mention/clearly imply use of theorem 8.1.

 0.5 f is differentiable on (-1,1)
 0.5 Its derivative is given by term wise differentiation. 1-x+x^2+…

0.5 f is continuous (being differentiable, but reasoning not necessary as long as theorem 8.1 is invoked/implied somewhere)

1 f is not uniformly continuous on because of problem near -1

some reason should be given e.g. because f'(x) = 1/(1+x) monotonically goes to infinity as x goes to -1 from right,

so uniform continuity will violate MVT near -1.

1 f is uniformly continuous on any subinterval [a,1) because the derivative is bounded.

(0.5 for saying f is uniformly continuous on any closed interval [a,b] with b < 1.)

Q9 total 7 points

2 for increasing

1 for derivative positive at some point using MVT

1 for all f' everywhere positive (using intermediate value property for derivative), so f increasing on the real line

5 for construction plus explanation showing that all required properties are satisfied. Not having explicit "formula" is ok

e.g. add antiderivative of a suitable function (with total area) finite to something like arctan

suitable function can be made out of pulses e.g. triangles with height 1 and base widths 1/2^i at each positive integer i

1 if they only show that if limit exists then it must be 0

1 more for indicating what a counterexample must do:

derivative must not have a limit at infinity, so there must be an epsilon > 0 such that there are arbitrarily large x with f(x) > epsilon.

(ok if they do not do above two steps but solve the problem)

(lose 1 if they don't ensure the function is strictly increasing)

Q10 (a) 1.5 (b) 1.5 (c) 1.5 + 2.5

(a) order immaterial (0.5) because absolutely convergent cf Q3 (1)

(b) because monotonic increasing (1.5)

attempt to directly show integrable using definition: correct setup gets nothing

if they say that inf/sup is at left/right end then give 0.5

rest depends on completing the argument from there

(c) 1.5 + 2.5 as follows.

discontinuous at x = a_i because the value there is less than every subsequent value by at least 0.2024^i and therefore f(ai+) - f(ai) is at least 0.2024^i (1.5)

continuous at any other x: given epsilon, find N such that the tail sum beyond N of the geometric series is less than epsilon. find delta interval around x excluding a_i corresponding to first N terms etc (2.5)