Analysis 1 Final

Announced corrections/explanations in blue

- Start each problem on a **NEW sheet of paper.** There should be no notes asking to look somewhere else. If you forget, at the end you will have to copy on a new sheet of paper. Photograph your answers before submitting. Upload a scan on moodle before leaving.
- Justify *everything* rigorously using definitions and theorems we saw. You may use any result we proved and any result in a previous problem (even if you did not do that problem), unless doing so renders the problem trivial. **Ask me in case of any doubt.** You may not use L'Hospital's rule or Lebesgue's integrability criterion except in the first question.
- All integrals are Riemann integrals (no Stieltjes), i.e., of type $\int_a^b f$ with no $d\alpha$ (i.e., $\alpha(x) = x$). If $a = b$, we define $\int_a^b f = 0$. If $a > b$, we define $\int_a^b f = -\int_b^a f$.

1. **Short questions. Explain briefly.**

   (a) For a bounded sequence $\{x_n\}$ of real numbers, let s_i = supremum of the tail $x_i, x_{i+1}, \ldots$. Then there is a convergent subsequence of $\{x_n\}$ whose limit is the infimum of all s_i. True or false?

   (b) Suppose for differentiable functions f and g on $\mathbb{R}$, the limit as $x \to 0$ of each function is 0, but the limit of $\frac{f'(x)}{g'(x)}$ does not exist. Then the limit of $\frac{f(x)}{g(x)}$ also does not exist. True or false?

   (c) If a bounded subset S of $\mathbb{R}^3$ contains all its limit points, then any sequence in S has a convergent subsequence whose limit is in S. True or false?

   (d) Calculate the linear Taylor approximation for $f(x) = x^{\frac{1}{3}}$ around $a = 1$ carefully explaining the error term. Does f have a linear Taylor approximation at any value of a in its domain?

   (e) Show using the Lebesgue criterion that if f and g are integrable on $[0, 1]$, then so are fg and $|f|$.

2. An interval is a set of the type $[a, b], (a, b), [a, b)$ or $(a, b]$ with $a < b$. Note that a is allowed to be $-\infty$ for type "$(a,$ " and similarly b is allowed to be ∞ for type " $, b)$". Let $f : X \to \mathbb{R}$ be a continuous function where X is an interval.

   (a) For $X = [0,1]$, must $f(X)$ be an interval? Must $f(X)$ be bounded? Can $f(X)$ be any one of the four types of bounded interval (for appropriate f)?

   (b) Now let X be any interval. Must $f(X)$ be an interval? Must $f(X)$ be bounded if X is bounded?

   (c) Now let X be an *unbounded* interval. Can $f(X)$ it be any one of the four types of bounded interval? As formulated, this requires 12 answers: four answers for each of the three types of given unbounded X. I changed the interpretation to require just four answers: for each of the four types of *bounded* intervals I, you had to find (if possible) a single unbounded interval X and an f with $f(X)$ of type I.

3. This is a short question despite appearances. Suppose $\sum_{i=1}^{\infty} |a_i|$ converges. So $\sum_{i=1}^{\infty} a_i$ converges, say to L. Suppose $\{b_i\}$ is a rearrangement of the sequence $\{a_i\}$. Formally, take a bijection $\sigma : \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ and define $b_i = a_{\sigma(i)}$. Complete the argument below to show that $\sum_{i=1}^{\infty} b_i$ also converges to L. You are (discouraged but) allowed to give your own proof instead, but it has to be perfect.

   Proof: Consider partial sums $s_k = \sum_{i=1}^{k} a_i$ and $t_k = \sum_{i=1}^{k} b_i$. We will show that given $\epsilon > 0$, one has $|s_k - t_k| < \epsilon$ for all sufficiently large k. Choose N such that for $m \geq n \geq N$, we have $\sum_{i=n}^{m} \quad \ldots\ldots \quad < \epsilon$ (fill in the blank suitably). Now choose p such that $\{1, 2, \ldots, N\} \subset \{\sigma(1), \ldots, \sigma(p)\}$, i.e., the fist N terms of the sequence $\{a_i\}$ are contained in the first p terms of the rearranged sequence $\{b_i = a_{\sigma(i)}\}$. Carefully show that for $k > p$, we have $|s_k - t_k| < \epsilon$ and complete the proof.

4. Suppose $f : \mathbb{Q} \to \mathbb{R}$ is a uniformly continuous function. In quiz 2, you saw an argument (sketched below) showing that f extends uniquely to a continuous function $g : \mathbb{R} \to \mathbb{R}$. Show that g is in fact uniformly continuous. You may use the terminology from the provided sketch.

   Sketch from Quiz 2: For a given real number x, fix a sequence $\{x_n\}$ in $\mathbb{Q}$ converging to x. So $\{x_n\}$ is Cauchy. So $\{f(x_n)\}$ is Cauchy (because ...). So $\{f(x_n)\}$ converges, say to z. This forces us to define $g(x) = z$. Now one shows that z is independent of the chosen sequence. *Do not fill this sketch! The question asks something else!*

5. Let

$$f_n(x) = \frac{\cos(nx)}{n^{2.024}}.$$

(a) Show that $f(x) = \sum_{n=1}^{\infty} f_n(x)$ defines a continuous function with domain $\mathbb{R}$.

(b) Is f differentiable? If so, calculate $f'(x)$.

(c) Is f integrable over $[0, \frac{\pi}{2}]$? If so, calculate the integral. Is f integrable over $[0, \infty)$?

6. (a) For a differentiable function f on $\mathbb{R}$, $|f'(x)| < M$ for all real x. Show that f is uniformly continuous.

(b) Produce a uniformly continuous differentiable g on $\mathbb{R}$ with g' unbounded.

7. (a) In highschool one often sees $\int_0^1 f(x)dx$ defined by partitioning $[0, 1]$ into n equal intervals by taking $x_i = \frac{i}{n}$ and taking the limit of "left hand Riemann sums" $\sum_{i=0}^{n-1} \frac{1}{n} f(x_i)$ as $n \to \infty$. (Or one can take the right hand sums. One could also require require the limit of both sums to be the same.) Show that for continuous f, this gives the same result as what we did (i.e., $\int_0^1 f(x)dx = \lim_{n\to\infty} \sum_{i=0}^{n-1} \frac{1}{n} f(x_i)$).

(b) Find a non-integrable function for which the school integral exists.

8. Consider the power series $x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n-1} \frac{x^n}{n} + \cdots$.

(a) Find the radius of convergence R.

(b) Define $f(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n-1} \frac{x^n}{n} + \cdots$ for $x \in (-R, R)$. Is f continuous? Uniformly continuous? If the answer is no in either case, is it yes on some subinterval?

(c) Is f differentiable? If the answer is no, is it yes for some subinterval? In that case what is the derivative (where it is defined)?

9. Let f be a differentiable function on $[0, \infty)$ with $f(0) = 0, f(1) > 0, \lim_{x\to\infty} f(x) = L$ *and* $f'(x)$ is never 0. (Example: $f(x) = 1 - e^{-x}$.) Is f necessarily increasing? Is it necessary that $\lim_{x\to\infty} f'(x) = 0$? Added during the exam: the answer to the second question is NO.

10. Suppose S is a countably infinite subset of $(0, 1)$, enumerated as $a_1, a_2, a_3, \ldots$.

(a) Show that $f(x) = \sum_i (0.2024)^i$, with the sum taken over $\{i \mid a_i < x\}$, is a function on $(0, 1)$, the point being to justify why is it well-defined. As usual the empty sum is defined to be 0.

(b) Show that f is integrable.

(c) Show that f is discontinuous exactly at $x \in S$ and continuous elsewhere.