HW 1

This HW has no analysis and contains exercises "only" in mathematical writing in the context of proving some foundational facts about sets and functions. But it is extremely important because (1) writing well is critically important and (2) these foundational facts are used routinely without any explanation. Results of exercises 1 and 2 should enter your bloodstream. They are also short work if you do them right. All parts of exercise 3 should become easy enough to do effortlessly whenever required.

0. Email me right now at least two sentences describing an interest of yours in something *other than* mathematics, computer science, physics or any theoretical science, subject to the rules below. Feel free to write a paragraph describing your emotional and/or intellectual relationship with this activity if you wish. I'm not so good at remembering names and quite bad in associating faces with names, though I do hope to get there in time. I am hoping that knowing something more about you will help me.

Rules. It should be something that you pursue or have pursued *actively* if not passionately. It can be any art, craft, sport, trade or hobby that you do outside of school requirements. It is not necessary that you excel at it, but I am looking for something in which you engage *thoughtfully, voluntarily*, and in which you have some *emotional investment*.

If your involvement consists primarily of partaking of some experience created by others, like reading, listening to music, watching films or a sport, then it should preferably be something you do with discernment in the sense that you have some knowledge or curiosity about the craftsmanship that goes into it.

Write solutions to all HW problems (in TEX) with extreme clarity and precision. Aim to *compose publishable solutions* from which someone can learn. See writing and solving tips given separately.

1. A function $f : S \to T$ is called *invertible* or an *isomorphism of sets* if there is a function $g : T \to S$, called *an* inverse of f, such that $f \circ g = id_T$ and $g \circ f = id_S$. Show that (a) f is invertible if and only if f is both injective (i.e., one-to-one) and surjective (i.e., onto), (b) if f is invertible then it has a unique inverse.

Note that being an *if and only if* statement, part (a) requires two arguments. One of the two is so immediate that it may be confusing to articulate it precisely. Try to do the rest too as concisely and incisively as you can. It may help to explore the consequence of just one of the two required equations and prove a subclaim. Does $f \circ g = id_T$ imply something about f? About g?

Optional extensions. (c) *Results mining the same vein.* A function $f : S \to T$ is called left invertible if it has a left inverse, i.e., a function $g : T \to S$ with $gf = id_S$. (We're dropping $\circ$ from now on.) Which functions have a left inverse? If a left inverse exists, must it be unique? Repeat for right inverses. What can you say about a function having a left inverse and a right inverse?

Now call f right cancellable if for any functions h_1, h_2, the equation $h_1 f = h_2 f$ implies $h_1 = h_2$. Can you see an immediate formal connection between being left/right cancellable and left/right invertible? Which functions are right cancellable? Left cancellable? Both?

Note: This part gets one's toes wet in formulating statements purely in terms of functions and their composition as opposed to using elements. Analogues of such statements and this style of doing business *might* become relevant for some of you later on when you deal with categories of objects other than plain sets. Unless and until that happens, don't get too fond of such things.

(d) *Two-out-of-three property.* See informally for yourself that any two of the following properties for a function between *finite* sets implies the third: injective, surjective, domain and codomain have the same finite cardinality. In particular a function between finite sets of equal cardinality is injective if and only if it is surjective. This is not so exciting for sets but its analogue for vector spaces is quite useful, as you will see. What happens if we drop the word *finite* everywhere?

2. Consider the commonly made statement "Defining an equivalence relation on a set S is the same as defining a partition of S". Make the preceding vague statement precise. Then prove the precise statement. Both underlined notions are defined below. Hint: I almost made this a part of the previous problem.

A *partition* of S is defined to be a set of pairwise disjoint subsets of S whose union is S, i.e., a partition of S is a set $\{S_\alpha | \alpha \in I\}$, where α ranges over some index set I and each $S_\alpha \subset S$, such that $\bigcup_{\alpha \in I} S_\alpha = S$ and for all distinct $\alpha, \beta \in I$, one has $S_\alpha \cap S_\beta = \varnothing$.

Recall that a relation on a set S means a relation from S to itself, i.e., a subset of $S \times S$. A relation $\sim$ on a set S is called an equivalence relation if it is

- reflexive (i.e., for each $x \in S$, one has $x \sim x$),
- symmetric (i.e., whenever $x \sim y$, one must also have $y \sim x$), and
- transitive (i.e., whenever $x \sim y$ and $y \sim z$, one must also have $x \sim z$).

As an aside, recall that changing *symmetric* to *antisymmetric* gives the notion of a partial order, a completely different kind of beast.

3. Let $f : S \to T$ be a function. Recall two definitions: for $A \subset S$, we have $f(A) = \{f(x) | x \in A\}$, a subset of T. For $B \subset T$, we have $f^{-1}(B) = \{x | f(x) \in B\}$, a subset of S. The latter notation does not require f^{-1} to be a function from T (but can be thought of as defining a function from the power set of T to that of S). Answer the following with proofs/counterexamples as necessary. While all questions are phrased as asking for a YES/NO answer, supply the strongest results that you can for all parts.

(a) Is $f(A_1 \cup A_2) = f(A_1) \cup f(A_2)$, i.e., does f preserve finite unions? (Why "i.e."?) Arbitrary unions?

(b) Does f preserve finite intersections? Arbitrary intersections?

(c) Does f preserve complements, i.e., is $f(A^c) = f(A)^c$? More precisely, is $f(S \setminus A) = T \setminus f(A)$?

(d) Repeat the previous questions for f^{-1}.

(e) Is $f^{-1}(f(A)) = A$? Is $f(f^{-1}(B)) = B$?

(f) For each NO answer, does it change to YES for some functions f other than isomorphisms?