Analysis 1 Test 1

Justify *everything* precisely. You may use any result we proved and any result in a previous problem (even if you did not do that problem), unless doing so renders the problem trivial. Ask me in case of any doubt.

Photograph your answers before submitting. Upload a scan on moodle before leaving.

1. **Short independent questions**

   (a) Let $f : X \to Y$ be a continuous function between metric spaces. Suppose X is compact. Show that $f(X)$ is a compact subset of Y. Deduce that if $Y = \mathbb{R}$, then $f(X)$ has a maximum and a minimum element. This is utterly standard so only a perfect proof will be accepted. You may use the following outline.

   Let $\{V_\alpha\}_{\alpha \in I}$ be an ... of ... , i.e., each set V_α is ... in ... and the union $\bigcup_{\alpha \in I} V_\alpha$ ...
   As f is continuous, we know that ...
   Now consider the family ... of subsets of ...
   We may apply the hypothesis on the metric space ... to this family because ...
   Therefore ...
   If $Y = \mathbb{R}$, let $M = \sup f(X)$ and $m = \inf f(X)$. Both of these exist because ...
   In fact $M \in f(X)$ and $m \in f(X)$ because ...

   (b) Show that if $\{x_n\}$ is a Cauchy sequence in $\mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ is a continuous function, then $\{f(x_n)\}$ is also a Cauchy sequence. Show that this is false in general if the domain is a proper subset of $\mathbb{R}$.

   (c) For a subset E of $\mathbb{R}$, it is given that for *each* continuous $f : E \to \mathbb{R}$, the set $f(E)$ is bounded in $\mathbb{R}$. Show that $f(E)$ must in fact be compact for any continuous f. (Hint: compose with something.) Must E itself be compact?

2. For a nonempty subset E of $\mathbb{R}$, define a function $f_E : \mathbb{R} \to \mathbb{R}$ by $f_E(x) = \inf_{p \in E} |x - p|$.

   (a) Sketch the graph of f_E when $E = \{0\} \cup (1, 2]$.

   (b) Fix a nonempty E. Prove that f_E is continuous. Begin as follows: Let $x \in \mathbb{R}$ and let $\epsilon > 0$.

   (c) Find the set of all x such that $f_E(x) = 0$. Your answer should be in terms of E.

   (d) Describe all subsets K of $\mathbb{R}$ such that for each nonempty $E \subset \mathbb{R}$, the set $f_E(K)$ is bounded above. Among these sets, find those K such that for each $E \subset \mathbb{R}$, the set $f_E(K)$ has a maximum element.

3. Let $A \subset \mathbb{R}$ be bounded. Suppose you are given a real number y and a sequence $\{p_n\}$ in A with the property that every convergent subsequence of $\{p_n\}$ converges to y. (a) Prove that $p_n \to y$. (b) Show that the result is false (even in a silly way) if the boundedness assumption is dropped.

4. Let X be a nonempty metric space. (a) Suppose $p_n \to p$ and $q_n \to q$. Show that $d(p_n, q_n) \to d(p, q)$. (b) Now assume X is compact. Show that there are points x, y such that $d(x, y) =$ the diameter of X.

5. (a) Let $K_0 \supset K_1 \supset \cdots$ be a nested sequence of nonempty compact sets in $\mathbb{R}$. Let $d_n =$ the diameter of K_n. Recall that $\bigcap_n K_n = K$ is nonempty. We know that if the sequence $d_n \to 0$ then $K = \{p\}$ for some $p \in \mathbb{R}$. Prove the converse by showing that if there is $e > 0$ such that each $d_n \geq e$, then the diameter of K is also at least e.

   (b) A function $f : \mathbb{R} \to \mathbb{R}$ has the following properties. Show that f is continuous.

   - For any compact set K in $\mathbb{R}$, the image $f(K)$ is compact.
   - For every nested sequence $K_0 \supset K_1 \supset \cdots$ of compact sets in $\mathbb{R}$ we have $f(\bigcap_n K_n) = \bigcap_n f(K_n)$.

6. Consider a function $f : [0, 1] \to \mathbb{R}$. The graph of f is the subset $\{(x, f(x)) | x \in [0, 1]\}$ of $\mathbb{R}^2$. Show:

   (a) If f is continuous then its graph is compact.

   (b) If the graph of f is compact, then f is continuous.