Analysis 1 Test 2

- Start each problem on a **NEW sheet of paper.** There should be no notes asking to look somewhere else. If you forget, at the end you will have to copy on a new sheet of paper. Photograph your answers before submitting. Upload a scan on moodle before leaving.
- Justify *everything* rigorously using definitions and theorems we saw. This needs added emphasis for the topics on this test, which you already "knew" before you set foot in CMI.
- You may use any result we proved and any result in a previous problem (even if you did not do that problem), unless doing so renders the problem trivial. **Ask me in case of any doubt.**
- All integrals are Riemann integrals (no Stieltjes), i.e., of type $\int_a^b f$ with no $d\alpha$ (i.e., $\alpha(x) = x$).

1. (i) Suppose $f : [0, 2] \to \mathbb{R}$ is differentiable with $f(0) = 0, f(1) = 1, f(2) = 3$. Prove that there exists an x with $f'(x) = \sqrt{2}$. What if f is allowed to be nondifferentiable at a single point (but still required to be continuous)?

   (ii) For $a < c < b$ show that f is integrable on $[a, b]$ if and only if it is integrable on both $[a, c]$ and on $[c, b]$. Under these conditions relate the three integrals.

2. Suppose f and g are continuously differentiable (i.e., i.e., f' and g' exist and are continuous) and strictly increasing functions from $\mathbb{R}$ to $\mathbb{R}$.

   (i) For $a < b$, show the following where you should replace both the ? with appropriate real numbers. Clearly state exactly which part of the hypotheses is used in each step.

   $$\int_a^b f(g(x))g'(x)dx = \int_?^? f(t)dt$$

   (ii) The proof stays valid for (much) weaker hypotheses. Give the weakest hypotheses that you can. Specifically, consider domain, differentiability and monotonicity of either function. Explain fully. Feel free to define $\int_p^q$ with $p \geq q$ if you must, but as always justify.

3. For $x > 1$, define $L(x) = \int_1^x \frac{1}{t}dt$. Note that this is NOT a function we have studied, so you should make no assumptions about it.

   (i) Calculate $\int_1^y L(x)dx$ where your answer may be in terms of L.

   (ii) Show the following inequality for $z \in (0, 1]$. Can the inequality be made strict?

   $$\left|L(1+z) - \left(z - \frac{z^2}{2} + \frac{z^3}{3} - \cdots + (-1)^{n-1}\frac{z^n}{n}\right)\right| \leq \frac{z^{n+1}}{n+1}.$$

4. Recall that a non-continuous function may be integrable (e.g., you saw that $f : [0, 1] \to \mathbb{R}$ defined by $f(0) = 1, f(x) = 0$ for $x \neq 0$ is integrable) or not (you saw an example in the HW) Do not prove any of this. It is only a reminder.

   (i) Thomae's function g is defined as follows on [0,1]. Definition: If x is irrational $g(x) = 0$. For rational input in reduced form $\frac{p}{q}$, define $g(\frac{p}{q}) = \frac{1}{q}$, e.g., $g(0.6) = \frac{1}{5}$. Calculate $\int_0^1 g$ from first principles.

   Hint: for any positive integer n, there are only finitely many rational numbers in [0,1] with $g(\frac{p}{q}) \geq \frac{1}{n}$ (all are in the list $0, 1, \frac{i}{j}$ with $0 < i < j \leq n$). These can be enclosed in disjoint intervals whose lengths add up to arbitrarily small total ϵ. Choose appropriate ϵ and bound the relevant sum by $\frac{?}{n}$ where ? is a constant.

   (ii) Show that composition of two Riemann integrable functions need not be Riemann integrable.

5. (i) Show that the curve $\gamma : [0, 2\pi] \to \mathbb{C}$ given by $t \mapsto \exp\left(2\pi it\sin(\frac{1}{t})\right)$ is not rectifiable. Partly informal reasoning is ok.

   (ii) Suppose $f : (x - \delta, x + \delta) \to \mathbb{R}$ and suppose $f''(x)$ exists. Show that the following limit exists.

   $$\lim_{h \to 0} \frac{f(x + h) + 2f(x - \frac{h}{2}) - 3f(x)}{h^2}$$