Analysis 1 Quiz 0 This one does not count! Name: Arjun Agarwal

Photograph your answers before submitting. Upload a scan on moodle when prompted.

1. (*Summer exam question with a minor variation.*) Suppose a differentiable function f from $\mathbb{R}$ to $\mathbb{R}$ has a local minimum at $(a, f(a))$. This means there are real numbers m and M such that (i) $m < a < M$ and (ii) $f(a) \leq f(x)$ for any $x \in [m, M]$. The proof of a standard result is sketched below. Complete it as instructed using the given options.

Proof: For sufficiently __1__ $h < 0$, it is given that $f(a+h)$ __2__ __3__.

Therefore for such h the quantity __4__ must be __5__ __6__.

By taking the limit of this quantity as $h \to 0$ from the appropriate side, we get that __7__ must be __8__ __9__.

A parallel argument for suitable positive values of h gives that __10__ must be __11__ __12__.

Combining both conclusions gives the desired result: __13__ __14__ __15__. Note that the mentioned limits exist because __16__.

(i) Write a sequence of 9 letters indicating the correct options to fill in the numbered blanks 1 to 9.

~~KEI~~ ~~A~~CJKEILEI B

(ii) Write a sequence of 7 letters indicating the correct options to fill in the numbered blanks 10 to 16.

LCI LG-I M

Options

A. small	B. large	C. $\geq$
D. $>$	E. $\leq$	F. $<$
G. $=$	H. $\neq$	I. 0
J. $f(a)$	K. $\frac{f(a+h)-f(a)}{h}$	L. $f'(a)$
M. f is differentiable	N. f is continuous	

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2. Consider the following calculation, where L'Hôpital's rule is used in the first step. Note that as $x \to 0$, values of $\cos(x^{-1})$ and $\sin(x^{-1})$ keep oscillating but stay bounded between -1 and 1.

$$\lim_{x\to 0} \frac{x^2 \sin(x^{-1})}{\sin x} = \lim_{x \to 0} \frac{2x\sin(x^{-1}) + (x^2)(-x^{-2})\cos(x^{-1})}{\cos x} = \lim_{x\to 0} \frac{2x \sin(x^{-1})}{\cos x} - \lim_{x \to 0} \frac{\cos(x^{-1})}{\cos x}.$$

Of the two limits in the last step, the first is 0 due to the factor $2x$ but the second does not exist because $\cos(x^{-1})$ keeps oscillating in $[-1, 1]$ as $x \to 0$. So the original limit does not exist. **Is this reasoning right?**

The reasoning is incorrect. We first point out that

$$\lim_{x\to 0} \frac{x^2 \sin(x^{-1})}{\sin(x)} = \lim_{x\to 0} \frac{x^2 \sin(\frac{1}{x})}{(x - \frac{x^3}{3!} + \dots)} = \lim_{x \to 0} \frac{x^2 \sin(\frac{1}{x})}{x - O(x^3)}$$

Turn over →

$\Rightarrow \lim_{x\to 0} x \sin(\frac{1}{x})$ which is 0 as $\sin(\frac{1}{x})$ oscillates between -1 and 1.

PTO

3. (*Entrance exam question expanded.*) Suppose f is a function whose domain is X and codomain is Y. It is given that $|X| > 1$ and $|Y| > 1$. No other information is known about X, Y and f. For each statement listed below, write the numbers of *all* correct options (and no incorrect options!) that apply to that statement.

Statements

a) For each x in X and for each y in Y it is true that $f(x) = y$. Answer: 3, 5   2

b) For each x in X, there exists y in Y such that $f(x) = y$. Answer: 4, 5, 6   1, 4, 6, 8

c) For each y in Y, there exists x in X such that $f(x) = y$. Answer: 9   5, 6

d) There exists x in X and there exists y in Y such that $f(x) = y$. Answer: 1, 4, 6, 8

e) For each x in X, there exists a unique y in Y such that $f(x) = y$. Answer: 4, 5, 6   1, 4, 6, 8

f) For each y in Y, there exists a unique x in X such that $f(x) = y$. Answer: 3, 4   3, 5

g) There exists a unique x in X and there exists a unique y in Y such that $f(x) = y$. Answer: 3, 4

h) There exists a unique x in X such that for each y in Y it is true that $f(x) = y$. Answer: 4, 2

i) There exists a unique y in Y such that for all x in X it is true that $f(x) = y$. Answer: 7, 8

Note: in the next two statements, the symbol $\forall$ stands for "for all"

j) $\forall\, x_1$ in X and $\forall\, x_2$ in X and $\forall\, y$ in Y, if $f(x_1) = f(x_2) = y$ then $x_1 = x_2$. Answer: 3, 4

k) $\forall\, y_1$ in Y and $\forall\, y_2$ in Y, and $\forall\, x$ in X, if $f(x) = y_1 = y_2$ then $y_1 = y_2$. Answer: 1, 4, 6, 8

Options

1. The statement is true.
2. The statement is false.
3. If the statement is true then f is one-to-one.
4. If f is one-to-one then the statement is true.
5. If the statement is true then f is onto.
6. If f is onto then the statement is true.
7. If the statement is true then f is constant.
8. If f is constant then the statement is true.
9. None of the above.

A:

the reason this doesn't work as $\frac{x^2 \sin(\frac{1}{x})}{\sin(x)}$

$= x^2 \times \frac{\sin(\frac{1}{x})}{\sin(x)}$ which is of the form

$0 \times \infty$ which is not indeterminate which

It is actually $\frac{0}{0}$ form.

is a condition required to apply

L-Hopital making the application [illegible]

$\lim \frac{f'(x)}{}$ [illegible]

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