Analysis 1 Quiz 1 Answers
Comments are welcome

1. Short questions.

(a) Write a sequence of *positive* real numbers whose limsup is as large as possible and whose liminf is as small as possible (here small and large are understood in extended reals). Specify the sequence clearly and state the values of limsup and liminf. You need not justify.

Many answers are possible, e.g., $2, \frac{1}{2}, 3, \frac{1}{3}, 4, \frac{1}{4}, 5, \frac{1}{5}, \ldots, n, \frac{1}{n}, \ldots$, with $\limsup = \infty$ and $\liminf = 0$.

(b) "$p_n \to a$" in $\mathbb{R}$ means the following sentence S: for every $\epsilon > 0$, there is a positive integer N such that for all integers $n > N$, one has $|p_n - a| < \epsilon$ (Complete the sentence by filling in the blank space.)

Now write a similarly precise definition of the statement "A given sequence $\{p_n\}$ does NOT converge in $\mathbb{R}$". No justification is required. The answer "the sentence S is false for every real number a" will get no credit (but could be a starting point for your thinking).

For each $a \in \mathbb{R}$ there is $\epsilon > 0$ such that for each positive integer N, there is an integer $n > N$ with $|p_n - a| \geq \epsilon$.

(c) Consider three possible properties of a given sequence of real numbers.

A. Cauchy B. bounded below C. monotonically decreasing

Write all implications of the types (i) X $\Rightarrow$ Y and of type (ii) (X and Y) $\Rightarrow$ Z, where X, Y, Z are distinct labels chosen from A, B, C. **Justify briefly**. You may simply quote relevant results. Do not write implications of type (ii) subsumed in those of type (i). No need to give counterexamples for invalid implications.

A implies B as every Cauchy sequence is bounded on both sides. (Direct proof is easy or, if you wish, use that Cauchy implies convergent, which implies bounded.) In addition, B and C together imply that the sequence is convergent, hence Cauchy.

2. For a sequence $\{p_n\}$ of real numbers consider the four statements below. State and prove all implications among them. A (precise) sketch is ok. Hint: this is not arduous. One of the implications involving lim sup was briefly discussed in last class, but you should not just quote it. Recall that $\limsup_{n\to\infty} p_n = \lim_{m\to\infty} s_m = \inf_m s_m$ because $s_1 \geq s_2 \geq \ldots$. Here s_m is the supremum (in extended reals) of the "tail" $\{p_m, p_{m+1}, \ldots\}$.

(i) For any real x, there exists a positive integer n such that $p_n > x$.

(ii) For any real x and any positive integer m, there is an integer n such that $n > m$ and $p_n > x$.

(iii) $\limsup_{n\to\infty} p_n = \infty$

(iv) $\{p_n\}$ has a subsequence $\{p_{n_k}\}$ such that $p_{n_k} \to \infty$.

All four are equivalent. Here is a sketch showing (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (i), but one should really work this out for oneself. There is little learning value in seeing someone else's detailed proof.

For (i) $\Rightarrow$ (ii) argue by contradiction by applying (i) to the maximum of x and $p_1, \ldots, p_m$.

For (ii) $\Rightarrow$ (iii), observe that by (ii), for any real x, the supremum of any tail is $> x$.

For (iii) $\Rightarrow$ (iv) imitate the argument in class: pick any p_{n_1}. The supremum of each tail must be ∞. (Why?) So from the tail p_m with $m > n_1$, pick $p_{n_2} > p_{n_1} + 1$, and so on.

Finally (iv) $\Rightarrow$ (i) is direct by definition of $p_{n_k} \to \infty$.

3. Carefully show from basic principles (namely that $\mathbb{R}$ is a complete ordered field) that for any given real number x, there is an integer n such that $n < x$. I do not want to see the following argument (even if you reproduce the proof of the Archimedean property of $\mathbb{R}$): "We know that there is an integer $m > -x$. So $-m < x$ and hence $n = -m$ works." You can of course imitate the proof of the Archimedean property.

Imitate the proof of Rudin 1.20a. If every integer n satisfies $n \geq x$ then x is a lower bound for the set $\mathbb{Z}$ of all integers, which is a nonempty proper subset of $\mathbb{R}$. Therefore $\mathbb{Z}$ has a greatest lower bound, say α, in $\mathbb{R}$. But then $\alpha + 1$, being greater than α, is not a lower bound for $\mathbb{Z}$. So $\alpha + 1 > m$ for some integer m. But this implies that $\alpha >$ the integer $m - 1$, contradicting that α is a lower bound for every integer.

4. Suppose $\{p_n\}$ is a Cauchy sequence in a *metric space* X such that a certain subsequence $\{p_{n_k}\}$ converges to $p \in X$. Show from first principles that $p_n \to p$. Write the answer to this question in the space below.

This is Rudin problem 3.20, which was on HW 3. The argument is standard and was also sketched in class.