Analysis 1 Quiz 1 This one counts! Name: ____________________

Photograph your answers before submitting. Upload a scan on moodle before leaving.

1. Short questions.

(a) Write a sequence of *positive* real numbers whose limsup is as large as possible and whose liminf is as small as possible (here small and large are understood in extended reals). Specify the sequence clearly and state the values of limsup and liminf. You need not justify.

(b) "$p_n \to a$" in $\mathbb{R}$ means the following sentence S: for every $\epsilon > 0$, there is a positive integer N such that for all integers $n > N$, one has (Complete the sentence by filling in the blank space.)

Now write a similarly precise definition of the statement "A given sequence $\{p_n\}$ does NOT converge in $\mathbb{R}$". No justification is required. The answer "the sentence S is false for every real number a" will get no credit (but could be a starting point for your thinking).

(c) Consider three possible properties of a given sequence of real numbers.

A. Cauchy B. bounded below C. monotonically decreasing

Write all implications of the types (i) X $\Rightarrow$ Y and of type (ii) (X and Y) $\Rightarrow$ Z, where X, Y, Z are distinct labels chosen from A, B, C. **Justify briefly**. You may simply quote relevant results. Do not write implications of type (ii) subsumed in those of type (i). No need to give counterexamples for invalid implications.

2. For a sequence $\{p_n\}$ of real numbers consider the four statements below. State and prove all implications among them. A (precise) sketch is ok. Hint: this is not arduous. One of the implications involving lim sup was briefly discussed in last class, but you should not just quote it. Recall that $\limsup_{n\to\infty} p_n = \lim_{m\to\infty} s_m = \inf_m s_m$ because $s_1 \geq s_2 \geq \dots$. Here s_m is the supremum (in extended reals) of the "tail" $\{p_m, p_{m+1}, \dots\}$.

(i) For any real x, there exists a positive integer n such that $p_n > x$.

(ii) For any real x and any positive integer m, there is an integer n such that $n > m$ and $p_n > x$.

(iii) $\limsup_{n\to\infty} p_n = \infty$

(iv) $\{p_n\}$ has a subsequence $\{p_{n_k}\}$ such that $p_{n_k} \to \infty$.

3. Carefully show from basic principles (namely that $\mathbb{R}$ is a complete ordered field) that for any given real number x, there is an integer n such that $n < x$. I do the not want to see following argument (even if you reproduce the proof of the Archimedean property of $\mathbb{R}$): "We know that there is an integer $m > -x$. So $-m < x$ and hence $n = -m$ works." You can of course imitate the proof of the Archimedean property.

4. Suppose $\{p_n\}$ is a Cauchy sequence in a *metric space* X such that a certain subsequence $\{p_{n_k}\}$ converges to $p \in X$. Show from first principles that $p_n \to p$. Write the answer to this question in the space below.

Answer 2 and 3 separately, each on a different sheet. Write your name on each sheet.

Answer question 4 below.