Analysis 1 Quiz 2 Name: ____________________

Photograph your answers before submitting. Upload a scan on moodle before leaving.

1. Answer and explain briefly. (a) Is every *rational function* with domain [0,1) uniformly continuous? (Of course the denominator polynomial is nonzero at each point of the domain.)

(b) Which *polynomial* functions from domain $(0, 1)$ to $\mathbb{R}$ are uniformly continuous?

(c) Is $f(x) = x^2$ with domain $\mathbb{R}$ uniformly continuous? Which polynomials are uniformly continuous (on $\mathbb{R}$)? (Partly informal reasoning ok as long as idea is clear.)

2. Let $I = [0, 1]$. (a) Let $f : I \to I$ be a continuous function. There must be $c \in I$ such that $f(c) = \sqrt{c}$.

To prove this we can apply the following theorem (below) to the following function: ________________

Theorem name (statement if you don't remember the name) and brief explanation:

(b) The claim in (a) is false if we remove a single point from the domain, i.e., for $g : I - \{p\} \to I$. Give a counterexample for a given $p \in [0, 1]$.

(c) Let $S = I - \{\frac{1}{2}\}$. Let $A = [0, \frac{1}{2})$ and $B = (\frac{1}{2}, 1]$. For each statement below, write one of the three options and *explain briefly*. **Options**: (i) True for the given reason (ii) True but for a different reason (iii) False.

- A and B are NOT separated because A and B have a common limit point, namely $\frac{1}{2}$.

- $S = A \bigcup B$ is a NOT valid separation of S because $A = [0, \frac{1}{2})$ is not an open set.

- By the theorem you used in part (a), for a continuous function $f : S \to \mathbb{R}$, its image set $f(S)$ is either an interval (possibly going to infinity on one/both sides) or an interval minus a single point.

Name: ____________________________

3. Let $f : X \to Y$ be a function between metric spaces. (a) What does the statement "f is NOT uniformly continuous" mean? Your answer should be a single precise sentence formed by using the following phrases in correct order (with appropriate choices and blanks filled). You may use appropriate English connectors like *and/such that/with* as necessary.

(there exists / for every) $p \in X$	(there exists / for every) $q \in X$	$d_X(p,q)$ ____ δ
(there exists / for every) $\epsilon > 0$	(there exists / for every) $\delta > 0$	$d_Y(f(p), f(q))$ ____ ϵ

(b) Suppose f is uniformly continuous and $\{x_n\}$ is a Cauchy sequence in X. Complete the following proof that $\{f(x_n)\}$ is a Cauchy sequence in Y.

Given $\epsilon > 0$, choose $\delta > 0$ such that ...

This is possible because ...

For the same ϵ, choose a positive integer N such that for integers ...

This is possible because ...

This value of N gives the desired result. (No further explanation necessary!)

(c) Suppose $f : [0,1) \to \mathbb{R}$ is uniformly continuous. Show as follows that f can be uniquely extended to a continuous function from $[0,1]$ to $\mathbb{R}$.

Proof: Take a sequence $\{x_n\}$ in [0,1) converging to ______. Now the sequence $\{f(x_n)\}$ is _________________

because ...

Therefore the sequence $\{f(x_n)\}$ of real numbers _________________. As we need f to be _________________,

we must define $f(1) =$ _________________ $= z$ (notation). So if a desired extension exists, it is ____________.

It is sufficient to prove the claim that $(*)$ if $\{y_n\}$ is any other sequence such that ...

then we must have ...

For this, consider the sequence $x_1, y_1, x_2, y_2, \ldots$ and see that this sequence is also _________________ Justify.

Therefore by previous reasoning, the sequence $f(x_1), f(y_1), f(x_2), f(y_2), \ldots$ _________________

Turn over $\rightarrow$

(3c continued) This proves the claim ($*$) by of the following reasoning (and thus completes the proof):

(d) Is the extension in part (c) necessarily uniformly continuous?