CHENNAI MATHEMATICAL INSTITUTE

B.Sc. Analysis-2

End-term Examination, 2025, Aug-Nov

100

Part A

The problems in this section are either already discussed in the class or a slight variation of that. Give the required proof completely with all details. You can attempt at most five problems and score up to a maximum of 50 marks from this section.

1. If $U \subseteq \mathbb{R}^n$ is open and connected, then show that U is path connected. 10

2. Assuming Weierstrass approximation theorem (that polynomials are dense in $C([0,1])$) prove the Stone-Weierstrass theorem for $C(X)$ (over $\mathbb{R}$), where X is a compact metric space. 15

3. Construct the cantor set. Show that it is closed, perfect, uncountable, nowhere dense and totally disconnected. 16

4. Answer (with proofs) whether the set of all real invertible matrices in $M_n(\mathbb{R})$ and the set of all complex invertible matrices in $M_n(\mathbb{C})$ are path connected ? 10

5. Let H be a separable Hilbert space. Show that an orthonormal set $\{e_i\}_{i\in I}$ (where I is countable) is maximal if and only if $x = \sum_{i\in I}\langle x, e_i\rangle e_i$ for all $x \in H$. 12

6. If a family of bounded linear operators $\{T_\lambda\}_{\lambda\in\Lambda}$ on a complete normed linear space X do not have a uniform bound, that is $sup_{\lambda\in\Lambda}\|T_\lambda\| = \infty$, then show that the set $S = \{x \in X : sup_{\lambda\in\Lambda}\|T_\lambda x\| = \infty\}$ is dense in X. 10

7. Using uniform boundedness principle prove the existence of a function whose Fourier series does not converge at a given point. 20

8. Let $K \subset L^2([-\pi,\pi])$ be a compact subset. Prove that for any given $\epsilon \geq 0$, there exists an $N \in \mathbb{N}$, so that for any $f \in K$, $|n| \geq N$, $|\hat{f}(n)| < \epsilon$. 10

9. Let $f \in C^0([0,2\pi])$ is continuously differentiable. Show that the Fourier series converges uniformly to f. 12

Part B

You may use any result proved in the class or in assignments. You can score up to a maximum of 75 marks from this section.

$|f^{-1}(x) - f^{-1}(y)| > K\epsilon$

1. Let $f : \mathbb{R}^n \mapsto \mathbb{R}^n$ be a continuous function satisfying $\|f(x) - f(y)\| \geq K\|x - y\|$ for some $K > 0$. Show that f is a homeomorphism onto $\mathbb{R}^n$. 20

2. Let $f, g \in C^0([-\pi, \pi])$ satisfy $f(t) = g(t)$ for $t \in (t_0 - \delta, t_0 + \delta)$, where $t_0 \in (-\pi, \pi)$ and $\delta > 0$. Show that the Fourier series for f and g either both converge at t_0 or both diverge at t_0. 15

3. Let

$$\Delta_f(r) = \sup_{s,t \in [-\pi,\pi], |s-t| < r} |f(s) - f(t)|.$$

Let $f \in C^0([-\pi, \pi])$ satisfies

$$\int_0^{\pi} \frac{\Delta_f(r)}{|r|} dr < \infty.$$

Then show that the Fourier series $\{S_n(f) : n \in \mathbb{Z}\}$ converges uniformly to f. 20

$\sup_{x \in [-\pi,\pi]} |S_n(f)(x) - f(x)|$

4. Show that

$$\frac{(3x^2 - \pi^2)}{12} = \sum_{n=1}^{\infty} (-1)^n n^{-2} \cos nx$$

for all $x \in [-\pi, \pi]$. 20

$\sum_{n=-n}^{n} \int_{-\pi}^{\pi} f(x) e^{-inx}$

5. Assuming (Part A, Problem 7), prove that there exists uncountably many continuous functions on $[0, 2\pi]$, whose Fourier series diverge on a dense G_δ subset of $[0, 2\pi]$. 20

6. Let $f \in C^0([0, 2\pi])$. For $a > 0$, define

$$(A_a f)(t) = \sum_{n=-\infty}^{\infty} e^{-a|n|} \hat{f}(n) e^{int}, \quad \forall t \in [0, 2\pi]).$$

Prove that $A_a f \mapsto f$ uniformly as $a \mapsto 0$. 30

$\int_{-\pi}^{\pi} f(x+t) D_n(t) - f(x)$

$\sum_{n=-\infty}^{\infty} (e^{-a|n|} - 1)$

$\int |S_n(f) - f| < \int_{-\pi}^{\pi} f(x) D_n(x-t)\, dt$

$\int_{-\pi}^{\pi} \frac{3x^2 - \pi^2}{12} \sin(nx)$

$\int_{-\pi}^{\pi} f(x-t) D_n(x)$

$x^2 \sin(nx)$, $\frac{-\cos(nx)}{n}$, $+2$, $\frac{-\sin(nx)}{n^2}$, $\sin(nx)/n^3$