$f_y = d(x,y)$

$f_z = d(x,z)$

CHENNAI MATHEMATICAL INSTITUTE

B.Sc. Analysis-2

Mid-term Examination, 2025, Aug-Nov

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Part A

The problems in this section are either already discussed in the class or a slight variation of that. Give the required proof completely with all details. You can score up to a maximum of 45 marks from this section.

1. Let (X,d) be a metric space and let $\mathcal{K}$ denotes the family of all non-empty bounded closed subsets of X. For $A, B \in \mathcal{K}$ let

$$\rho(A,B) = \inf\{\epsilon > 0 : A \subseteq N_\epsilon(B) \text{ and } B \subseteq N_\epsilon(A)\},$$

where $N_\epsilon(A) = \{x \in X : d(x,A) < \epsilon\}$. Show that $(\mathcal{K}, \rho)$ is a metric space. (ρ is called the Hausdorff metric.) 15

2. Let K be a compact subset of a metric space X. Let $\{B_n\}_{n=1}^{\infty}$ be an open cover for K. Show that there exists an $\epsilon > 0$ such that any open ball of radius ϵ centered at any point $x \in K$ is contained in one of the B_ns. 10

3. Show that every metric space can be isometrically embedded in its space of bounded uniformly continuous real functions. $X \to C_b(X)$ 10

4. Let X be a compact metric space. Let $T : X \mapsto X$ satisfies

$$d(Tx, Ty) < d(x,y), \text{ if } x \neq y.$$

Show that T has a unique fixed point. 10

5. Prove that a uniformly bounded, equi-continuous subset of $C([0,1])$ is totally bounded. 15

6. For $f : [0,1] \mapsto \mathbb{R}$, define

$$(D^+ f)(a) = \limsup_{x \mapsto a^+} \frac{f(x) - f(a)}{x - a}.$$

Prove that for each $a \in [0,1]$ the set $\{f \in C[0,1] : (D^+ f)(a) = \infty\}$ is a dense G_δ subset. (A set is said to be G_δ if it is countable intersection of open sets.) 20

Part B

You may use any result proved in the class or in assignments. You can score up to a maximum of 75 marks from this section.

1. Prove that the function $f(x) = \sum_{n=1}^{\infty} \frac{x}{n^2} \sin nx$ is a well-defined continuous function on $[0, 1]$. 10

2. Show that the vector spaces $c_0 = \{\{x_n\}_{n=1}^{\infty} : x_n \mapsto 0\}$ and $c = \{\{x_n\}_{n=1}^{\infty} : x_n \in \mathbb{C},\ \{x_n\}_{n=1}^{\infty} \text{ is convergent}\}$ are complete with respect to $\|\cdot\|_\infty$. 10

3. Let $X = \mathbb{R}^2$ and d_2 be the usual metric on X. Define for $x, y \in X$,

$$\begin{aligned} \rho(x,y) &= d_2(x,y), \quad \text{if } x = \lambda y \text{ for some } \lambda \in \mathbb{R}, \\ &= d_2(x,0) + d_2(0,y), \quad \text{otherwise.} \end{aligned}$$

Show that the closed unit ball (in the metric d) is not compact in (X, ρ). Is it separable? 20

4. Let X be a metric space and $S \subseteq X$. A point x is called an accumulation point of a S if $(B(x,r) \setminus \{x\}) \cap S$ is non-empty for all $r > 0$. A set is called perfect if it is equal to all its accumulation points. Show that a compact perfect set is always uncountable. 20

5. Prove that the closure of $\{g : g(x) = \int_0^x f(t)dt,\ f \in C([0,1]),\ \|f\| \leq M\}$, where $M > 0$, is compact in $C([0,1])$. 10

6. Let (X, d) be a complete metric space, $T : X \mapsto X$ be a contraction mapping, and $x_0 \in X$ the fixed point of T. Let $\{\epsilon_n\}$ be a sequence of positive numbers converging to 0. Let $y_0 \in X$ and y_n satisfies $d(y_{n+1}, T(y_n)) \leq \epsilon_n$ for all n. Then show that $\lim_{n\to\infty} y_n = x_0$. 25

7. Show that the space of all non-empty compact subsets of a separable metric space M endowed with the Hausdorff metric (see problem 1, part A for definition) is separable. 30