Analysis 2 Quiz 1

13 September 2025

Problems 1 and 2 are worth 12 points each. Each subpart of problem 3 is worth 4 points. You will be marked out of 30. All the Best!

1. Prove that l^2, the space of all sequences $\{x_n\}_{n=1}^{\infty}$ of real numbers such that $\sum_{n=1}^{\infty} |x_n^2| < \infty$ is complete under the l^2 norm. Recall that l^2 norm $\|.\|_2$ is defined as follows:

   For any seqeunce $x := \{x_n\}_{n=1}^{\infty}$ in l^2 , $\|x\|_2 := \left(\sum_{n=1}^{\infty} |x_n|^2\right)^{\frac{1}{2}}$

2. Does there exist a uniformly continuous function $f : [1, \infty) \to \mathbb{R}$ such that $\sum_{n=1}^{\infty} \frac{1}{f(n)}$ converges?

   (Hint: Try to bound $|f(n+1) - f(n)|$)

   $\exists\ \delta\ x - y < \delta$
   $\Rightarrow f(x) - f(y) < \varepsilon$

3. A metric space (X, d) is said to be *second countable* if there exists a countable collection $\mathcal{B} = \{B_n\}_{n=1}^{\infty}$ of open subsets of X such that for every $x \in X$ and every open set U of X containing x, one has $x \in B_n \subset U$ for some $B_n \in \mathcal{B}$. (For example, $\mathbb{R}^n$ is second countable, with a countable basis being the collection of balls of positive rational radius centred at points with rational co-ordinates.)
   A point p in a metric space X is said to be a *condensation point* of a set $E \subset X$ if every open neighbourhood of p contains uncountably many points of E. (For example, 0 is a condensation point of $(0, 1) \subset \mathbb{R}$ but not of $\left\{\frac{1}{n} \mid n \in \mathbb{N}\right\}$. Note that condensation points are limit points.)

   Let X be a second countable metric space with countable basis $\mathcal{B} = \{B_n\}_{n=1}^{\infty}$ and $E \subset X$. (If you're uncomfortable working with a second countable metric space, you may take $X = \mathbb{R}^n$ throughout or in any subpart, at the modest expense of half a point taken off each subpart in which you assume $X = \mathbb{R}^n$.)

   (a) Let W be the union of those B_n for which $E \cap B_n$ is at most countable and let $P = W^c$ (P may be empty). Prove that P is the set of all condensation points of E, with at most countably many points of E not in P.
   (Intersecting P with E and taking $X = \mathbb{R}^n$, one obtains as a corollary, that an uncountable subset of $\mathbb{R}^n$ contains uncountably many condensation points. This gives an overkill of a proof of the result that an isolated subset of $\mathbb{R}$ is at most countable.)

   (b) Show that P is closed in X and therefore perfect. As a result, if E is closed, show that $P \subset E$, hence every closed subset of a second countable metric space is expressible as a disjoint union of a perfect set and a set which is at most countable.

   (c) Now, assume further that X is complete and let E be a closed subset of X. Let C and P be disjoint sets that are at most countable and perfect respectively, such that $E = C \cup P$. Prove that every point of P is a condensation point of E.

Please turn over

$B_k \cap E$ is c

$B_k^c \cap E$

$\bar{P}$ $\bar{P} - P \cap E$

$P \subset E$

$E \cap B_k$ −

(Hint: assume for the sake of contradiction that some $x \in P$ is not a condensation point of E. Then there exists an open neighbourhood U of x containing countably many points $\{x_n\}_{n=1}^{\infty}$ of E. WLOG, let $x_1 = x$. Take a small enough open ball U_2 around x_2 whose closure lies entirely inside U and does not contain x, and pick the smallest index n such that x_n lies in U_2. Repeat this procedure ensuring that the radii of the balls go to 0 and consider the subsequence of $\{x_n\}_{n=1}^{\infty}$ thus obtained. Why does this lead to a contradiction?)

(d) Finally, show that every condensation point of E is contained in P, hence P is the set of condensation points of E. This shows that a closed subset of a complete, second countable metric space is *uniquely* expressible as a disjoint union of a perfect set and a set that is at most countable. This is known as the Cantor Bendixson Theorem. Find a counterexample to unique expressibility if X is not assumed to be complete.

Bonus question, not for credit: How much of the above can be generalised to (second countable) topological spaces? What modifications, if any, would one need to make to the hypotheses?