Analysis 2 Quiz 2

12 October 2025

Each problem is worth 20 points. You may score a maximum of 40 points. All the best!

1. Let X be compact metric space and $C(X)$ is the ring of real continuous functions on X. Let J be a closed ideal in $C(X)$ (closed in sup-norm metric). If $f \in C(X)$ and f vanishes on the zero set of J, then show $f \in J$. zero set of J means $Z(J) = \{x \in X \,:\, h(x) = 0 \,\forall\, h \in J\}$ (Hint : Find a function $g \in J$ which is strictly positive outside the open set $\{x \in X \,:\, |f(x)| < \epsilon\}$, further what can we say about $\dfrac{nfg}{1+ng}$ for large n ?)

2. This question has two parts. Part a is for credit and Part b is not for credit and maybe hard so don't give much time on it.

   Definition Let $f : [0,1] \to \mathbb{R}$ be a function. We say that f is **Non monotonic** if and only if for any $a, b \in [0,1]$, f is *not monotonic* on $[a, b]$

   a: Show that there exists
   $$f : [0,1] \to \mathbb{R}$$
   such that f is Non-monotonic.

   Hints:

   (a) $C[0,1]$ is complete under the sup-norm.

   (b) Find properties of the set
   $$S = \{f \in C[0,1] \,|\, f \text{ is monotone in } [p, q]\},$$
   where $p, q \in \mathbb{Q} \cap [0,1]$,

   (c) Apply Baire Category Theorem

   b: Why taking any nowhere differentiable function in C[0,1] is example of such a function

3. For a metric space X, let $C(X)$ denote the space of all bounded continuous real (or complex) valued functions on X. In this problem, we characterise compact subsets of common metric spaces and recover[1] assorted results.

   (a) We first get the following result out of the way: Let X be compact. Prove that a subset $\mathcal{F} \subset C(X)$ is bounded and equicontinuous if and only if its closure $\overline{\mathcal{F}}$ is compact.

   (b) Let $X = \{1, \ldots, n\} \subset \mathbb{R}$. Then $C(X) \simeq \mathbb{R}^n$, with the isomorphism given by $f \mapsto (f(1), \ldots, f(n))$. Show that this is a norm preserving linear homeomorphism when $\mathbb{R}^n$ given _____ norm (fill in the blank!). This isomorphism enables us to speak of equicontinuity of subsets of $\mathbb{R}^n$.

[1] perhaps in a highly convoluted manner

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(c) Which subsets of $\mathbb{R}^n$ are equicontinuous? Use this to characterise compact subsets $\mathbb{R}^n$. You should recover a well-known result.

(d) Now, let $X = \{\frac{1}{n} | \ n \in \mathbb{N}\} \cup \{0\}$. Denote the space of convergent real sequenc by $Conv(\mathbb{R})$. Show that $C(X) \simeq Conv(\mathbb{R})$. As in the previous parts, this is a nor preserving linear homeomorphism when $Conv(\mathbb{R})$ is given the ____ norm (fill in t blank!). By means of this isomorphism, we may speak of equicontinuity of subsets $Conv(\mathbb{R})$.

(e) Which subsets of $Conv(\mathbb{R})$ are equicontinuous? Use this to characterise compact subse of $Conv(\mathbb{R})$.

(f) For a sequence S in $Conv(\mathbb{R})$, let $S(n)$ denote its n^{th} term. Using the above or otherwis prove the following:
Let $\{S_n\}_{n=1}^{\infty}$ be a countable subset of $Conv(\mathbb{R})$ such that the tails of all sequenc uniformly get ϵ-close for each $\epsilon > 0$, i.e. for any $\epsilon > 0$ there exists $N \in \mathbb{N}$ (depending ϵ) such that $|S(m) - S(n)| < \epsilon$ for all $S \in \{S_n\}_{n=1}^{\infty}$ whenever $m, n \geq N$. Then $\{S_n\}$ h a uniformly convergent subsequence (the subsequence must converge in the ____ nor from part d).