Analysis 2 Quiz 3

25 October 2025

Duration: 2 hours 15 minutes

This is a timed quiz. Problem 1 is worth 16 points. Problems 2 and 3 are worth 12 points each. You will be marked out of 30. All the best!

1. Let X be a normed linear space over $\mathbb{R}$. Then following holds

$$X \text{ is locally compact} \iff X \text{ is finite dimensional.}$$

Prove ($\Longrightarrow$) direction

Hints:

(a) Show that closed unit ball in X is compact.

(b) Prove that If M be a proper closed linear subspace of X. Then for each $\varepsilon > 0$, there exists $x \in X$ such that

$$\|x\| = 1 \quad \text{and} \quad d(x, M) > 1 - \varepsilon,$$

where

$$d(x, M) = \inf_{y \in M} \|x - y\|.$$

2. If f is continuous on $[0,1]$ and if

$$\int_0^1 f(x)x^n\,dx = 0 \quad (n = 0, 1, 2, \dots)$$

prove that $f(x) = 0$ on $[0,1]$.

$||T(x,y)|| = ||T_x(y)|| < M_x ||y||$

3. Suppose X is a Banach Space, and $T : X \times X \to X$ is a bilinear map (linear in each coordinates). Suppose for each $x \in X$, the linear map $T_x(y) = T(x,y)$ is a bounded linear operator on X, also for each $y \in X$, $T^y(x) = T(x,y)$ is also a bounded linear operator. Prove that there is some $M > 0$ such that $||T(x,y)|| \leq M||x||\,||y||$.

T^y

$|T_x(y)| <$

$f_n = \sum_{k=0}^{n} f(\frac{k}{n})\binom{n}{k} x^k(1-x)^{n-k}$

4. The following problem is NOT for credit. It was supposed to be on the quiz but was inadvertantly shared yesterday after the tutorial. I am adding it for the sake of completeness.

Prove that every C^k function f on $[0,1]$ can be uniformly approximated by polynomials $\{p_n\}_{n=1}^{\infty}$ such that the ith derivative $f^{(i)}$ is uniformly approximated by $\{p_n^{(i)}\}_{n=1}^{\infty}$ for each $0 \leq i \leq k$. (Hint: what can you say about integrals of a uniformly convergent sequence of functions? This should suggest a promising location to start approximating from)

Comments and supplementary exercises (to be read after the quiz):

(a) Verify (at your leisure) that the space $C^k([0,1])$ of all C^k functions on $[0,1]$ is a Banach space under the norm

$$\|f\|_{C^k} := \sum_{i=0}^{k} \|f^{(i)}\|_\infty$$

and the topology induced by $\|.\|_{C^k}$ is strictly finer than the subspace topology inherited from $C([0,1])$, i.e. covergence under $\|.\|_{C^k}$ implies convergence under the sup norm but not necessarily vice versa. Find a sequence of C^k functions converging uniformly to a C^k function under the sup norm but not under $\|.\|_{C^k}$.

Problem 4 says that polynomials are dense in $C^k([0,1])$ under $\|.\|_{C^k}$ norm.

(b) The space $C^\infty([0,1])$ of C^∞ functions on $[0,1]$ is a complete metric space under the metric

$$d(f,g) := \sum_{n=0}^{\infty} \frac{1}{2^n} \min(1, \|f^{(n)} - g^{(n)}\|_\infty)$$

Why is $C^\infty([0,1])$ not a Banach space, i.e why does the metric d not come from a norm? $C^\infty([0,1])$ is a *Frechet space* (look this up if you're interested). Prove that polynomials are dense in $C^\infty([0,1])$. Can every $f \in C^\infty([0,1])$ be uniformly approximated by polynomials $\{p_n\}_{n=1}^{\infty}$ such that $f^{(k)}$ is uniformly approximated by $\{p_n^{(k)}\}_{n=1}^{\infty}$ for all $k \in \mathbb{Z}_{\geq 0}$?