CALCULUS II, Final Exam

26/11/2025, 9:30–12:30. Total Marks: 80

1. (10 points) Integrate $x^2 + y^2$ over the region $A \subset \mathbb{R}^2$ in the first quadrant bounded by $xy = 1, xy = 4, y = x, y = 2x$. Hint: Use a change of variables. $\frac{315}{16}$

2. (10 points) Show that $\mathbb{R}^2 \setminus \{0\}$ is not diffeomorphic to $\mathbb{R}^2$. ¿¡

3. (10 points) Let n be a positive integer. An *n-surface* is a non-empty subset S of $\mathbb{R}^{n+1}$ of the form $S = f^{-1}(c)$ where $f : \mathbb{R}^{n+1} \to \mathbb{R}$ is a C^∞-function such that $Df(p) \neq 0$ for all $p \in S$ and $c \in \mathbb{R}$. Let S be an n-surface and let $p \in S$. The tangent space $T_p(S)$ of S at p is defined as:

$$T_p(S) = \{v_p \in T_p(\mathbb{R}^{n+1}) \mid Df(p) \cdot v = 0\}.$$

$ab - cd = 1$

In the above definition, $Df(p)$ is considered as a vector in $\mathbb{R}^{n+1}$ and $Df(p) \cdot v$ denote the dot product.

   (a) Let $SL_2(\mathbb{R})$ denote the set of 2×2 real matrices with determinant 1. Using the natural inclusion $SL_2(\mathbb{R}) \subset \mathbb{R}^4$, show that $SL_2(\mathbb{R})$ is a 3-surface.

   (b) Let $p \in SL_2(\mathbb{R})$ denote the identity matrix. What is the tangent space $T_p(SL_2(\mathbb{R}))$? Describe it as a set of matrices with a certain property. trace 0

4. (a) (10 points) Let ω be a 1-form on an open set $U \subset \mathbb{R}^n$ such that $\int_\gamma \omega = 0$ for every closed[1] 1-cube γ in U. Prove that ω is exact on U, that is there exists $f \in C^\infty(U)$ such that $\omega = df$.

   (b) (10 points) Let ω be a closed 1-form on $\mathbb{R}^3 \setminus \{0\}$. Prove that ω is exact. ?!

5. (15 points) Let $S \subset \mathbb{R}^3$ be the hyperboloid $x^2 + y^2 - z^2 = 1, -1 \leq z \leq 1$ oriented with outward normal. Calculate the following integral: $z = \sqrt{t^2-1}$

$$\int_S \frac{x dx \wedge dz}{\sqrt{x^2+y^2}} + \frac{y dy \wedge dz}{\sqrt{x^2+y^2}}.$$

$(t, \theta) \mapsto (t\cos\theta, y\sin\theta, \sqrt{t^2-1})$

Hint: Using cylindrical coordinates $x = r\cos\theta, y = r\sin\theta$ or otherwise, show that the integrand form is zero on S.

$0 - \sqrt{3}$ $\quad -2 \leq z \leq 1$

6. (15 points) Let $S \subset \mathbb{R}^3$ be the part of the sphere $x^2 + y^2 + z^2 = 4$ below the plane $z = 1$, oriented with upward normal. Calculate the integral $\int_S z dy \wedge dz$. $t^2 + z^2 = 4$ $\quad yz\,dz$ $\quad -\sqrt{3}\pi$

Hint: Use Stokes' theorem: find a 1-form ω such that $d\omega = z dy \wedge dz$ and integrate on the boundary of S.

π $\quad \sqrt{3} < t < \sqrt{3}$ $\quad 0 < \theta < 2\pi$ $\quad -\frac{z^2}{2} dy$

3π $\quad f_1(x,y,z) e_1^v + f_2(x,y,z) e_2^v + f_3(x,y,z) e_3^v$ $\quad (a, b, c) \mapsto (a, b, c, \frac{ab-1}{c})$

$ab - cd$

$xy = 4$ $\quad bda + adb$

$y/x = v$ $\quad -ddc$ $\quad -cdd$

68.75

0.3125

$16 \overline{)50}$

-48

20

-16

40

-32

80

$(0,1)$ $\quad (1,1)$ $\quad 0\ 0,0$

$t^2 - 1 = y^2$

$t^2 - y^2 = 1$

[1]that is, $\gamma(0) = \gamma(1)$.