

CALCULUS II, Mid-Semester Exam
9:30-12:30, 01/10/25, total marks: 60

1. (4x5=20 points) Determine whether the following assertions are true or false. Provide a brief justification for your answers.

- (a) If a set $C \subset \mathbb{R}^3$ has measure 0 then C is bounded or countable. *f*
- (b) Let $C \subset \mathbb{R}^n$ be a bounded set of measure zero and let A be a closed rectangle containing C . Then the integral $\int_A \chi_C$ always exists.
- (c) Let $A \subset \mathbb{R}^n$ be a bounded set. Suppose that the upper integral $\overline{\int}_A \chi_A$ is equal to 0. Then A has measure 0. *f*
- (d) The function $f: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ given by $f(x, y) = (x^2 + y, x^2 + y^2)$ is locally injective¹ on $X := \{(x, y) \in \mathbb{R}^2 \mid x \neq 0 \text{ and } y \neq \frac{1}{2}\}$.

2. (10 points) Let A be the region in $\mathbb{R}^2$ bounded by the functions $y = 1 - x^2$ and $y = x^2 - 3$. Evaluate the integral

$$\int_A x(y-1).$$

3. (10 points) Let

$$I = \int_{-1}^1 \int_{|y|}^2 (x+y)^2 dx dy.$$

- (a) Sketch the region in $\mathbb{R}^2$ where the integral I is defined.
- (b) Change the order of integration in I .
- (c) Find the value of I .

4. (10 points) Let $A = \{(x, y) \in \mathbb{R}^2 \mid 0 < x < 3 \text{ and } 0 < y < x^2\}$. Define a function $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ by

$$f(x) = 0, \text{ if } x \notin A \text{ and } f(x) = 1, \text{ if } x \in A.$$

- (a) Is f continuous at $(0, 0)$?
- (b) Show that $D_u f(0, 0)$ exists for all $0 \neq u \in \mathbb{R}^2$.
- (c) Evaluate the integral $\int_A f$.

5. (10 points) Let $f: [0, 1] \times [0, 1] \rightarrow \mathbb{R}$ be continuous. Suppose that $D_2 f = \partial f / \partial y$ exists and is continuous. For $0 \leq y \leq 1$, define

$$F(y) := \int_0^1 f(x, y) dx.$$

Show that $F'(y) = \int_0^1 D_2 f(x, y) dx$.

(Hint: Use Fubini's theorem and the fundamental theorem of calculus.)

¹ f is locally injective at a point $a \in X$ if there is an open neighbourhood U of a in X such that the restriction of f to U is injective.