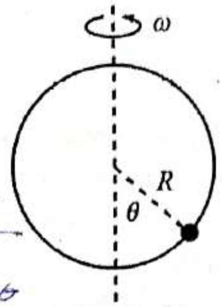
CM 1 – End-Sem-Exam – 29 Nov 2024, 9:30 am to 12:30 pm

1. {25 marks} **Bead on a rotating hoop.** A bead is free to slide along a frictionless hoop of radius R. Gravity acts downwards. The hoop rotates with constant angular speed ω around a vertical diameter (see Fig. 1). Find the equation of motion for the angle θ shown. What are the equilibrium positions? What is the frequency of small oscillations about the stable equilibrium? There is one value of ω that is rather special; what is it, and why is it special?



$\omega^2 R \cos\theta$

$= \sqrt{\frac{g}{R\cos\theta}}$

$\omega^2 R \sin\theta$

$mg = mR^2\omega^2\cos\theta$

$\sqrt{\frac{g}{R^2\cos\theta}}$

$\omega =$

$x = l\sin\theta$

$y = l(1-\cos\theta)$

$\dot{x} = l\cos\theta\,\dot\theta$

$\dot{y} = l\sin\theta\,\dot\theta$

$mg\,l(1-\cos\theta) + \frac{1}{2}ml^2\dot\theta^2$

$mg\,l\sin\theta$

$= -ml^2\ddot\theta$

$\sqrt{\frac{g}{l}}$

Figure 1: *Bead on a rotating hoop.*

2. {25 marks} **Decays**

   (a) A stationary mass M decays into a particle and a photon. If the speed of the particle is v, what is its mass? What is the energy of the photon? $\frac{Mv}{1+\beta}$

   $M\sqrt{\frac{1-\beta}{1+\beta}}$

   (b) A mass m moves at speed v. It decays into three photons, one of which travels in the forward direction, and the other two of which move at angles of 120° (in the lab frame) as shown in Fig. 2. What are the energies of these three photons?

$\frac{1}{\sqrt{1-\beta^2}}\,mv$

$\frac{m}{c}\frac{\sqrt{1+\beta}}{\sqrt{1-\beta}}$

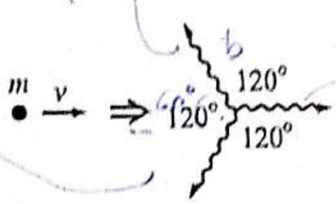
$\frac{g}{R}\cos\theta$

$-\frac{g}{l}\,t\,dt = \sin\theta\,d\theta$

$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = 2 \qquad \frac{1}{4} = 1-\frac{v^2}{c^2}$

$\frac{3}{4} = \frac{v^2}{c^2} \Rightarrow \sqrt{\frac{3c^2}{4}}$

$\frac{mc^2}{2}\frac{1+3\beta}{\sqrt{1-\beta^2}}$



$\frac{mc^3}{2}\sqrt{\frac{1+\beta}{1-\beta}}$

Figure 2: *Three photons decay.*

3. {25 marks} **Bullets on a train.**

$\frac{L}{v}$ A train moves at speed v. Bullets are successively fired at speed u (relative to the train) from the back of the train to the front. A new bullet is fired at the instant (as measured in the train frame) the previous bullet hits the front. In the frame of the ground, what fraction of the way along the train is a given bullet, at the instant (as measured in the ground frame) the next bullet is fired? What is the maximum number of bullets that are in flight at a given instant, in the ground frame? $\lceil \gamma \rceil$

in train frame $T = \frac{L}{u+v}$, $T = \frac{L}{vu}$

4. {25 marks} **Skimming a planet.** A particle travels in a parabolic orbit in a planet's gravitational field and skims the surface at its closest approach. The planet has mass density ρ. Relative to the center of the planet, what is the angular velocity of the particle as it skims the surface? $2R^2\sqrt{\frac{2}{3}G\rho\pi}$

$x = \gamma(x' + t'v)$

$\cancel{y = \gamma(y' + t}$

$t = \gamma(t' + x'v)$

$t_{total} = \frac{L}{u}$

$\frac{L}{v}$ $\qquad \frac{L}{u-v}$

$\xleftarrow{\quad L \quad}$

$\frac{L}{u}\cdot\frac{(u+v)}{v}$

$\frac{L}{u}v$ $\qquad L\,\frac{\frac{u+v}{1+uv}}{uv}$

$\frac{L(1+\frac{v}{u})}{v}$

$\boxed{\frac{L}{v}}$ $\qquad \frac{L}{v}$

$\frac{1}{\sqrt{1-v^2}}$

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