Some classic problems in discrete math

Problem 1 (Source: Laci Babai's puzzles set)**.** Given 13 integers, prove that if any 12 of them can be partitioned into two sets of six each with equal sums, then all the integers must be the same.

Problem 2 (Knaster's fixed point theorem, Putnam 57)**.** Let S be a set and P the set of all subsets of S. $f : P \mapsto P$ is such that if $X \subseteq Y$, then $f(X) \subseteq f(Y)$. Show that for some K, $f(K) = K$.

Problem 3 (Celebrity problem)**.** At a party, the celebrity is the person who knows nobody but everybody knows the celebrity. We would like to determine if there is a celebrity, if yes, who the celebrity is (obviously, there can be at most one celebrity in a group). We would be asking: does person x know person y? Show that we can determine if there is a celebrity in a party of n persons by asking only $3(n-1)$ questions. Show that we need to ask $2(n-1)$ questions in the worst case.

Problem 4 (Gossip and telephones)**.** There are n persons in a society, and each of them knows some item of gossip not known to the others. They communicate by telephone, and whenever one person calls another, they tell each other all that they know at that time. How many calls are required before each person knows everything?

Problem 5 (King in tournament)**.** Let us consider a complete directed graph/tournament (between every pair of vertices, there exists exactly one directed edge). A king in a tournament is a vertex v such that every other vertex is reachable from v via a directed path of length at most 2. Prove that in any tournament there is at least one king. There are at least three different ways to prove this.

Problem 6. A library has n books and $n+1$ subscribers. Each subscriber read at least one book from the library. Prove that there must exist two disjoint sets of subscribers who read exactly the same books (that is, the union of the books read by the subscribers in each set is the same). Hint: Very basic linear algebra.

Problem 7. We are vacationing in Graph National Park, which has n landmarks (nodes) and m trails (undirected edges) connecting pairs of landmarks. Each trail has a difficulty rating (weight). We would like to hike as many trails as possible, without repeating any trails. However, we will get increasingly tired as we hike, and so we are only willing to hike trails in nonincreasing order of difficulty. That is, after we hike a trail t_i, our next trail t_{i+1} must depart from the endpoint of t_i and its difficulty rating may not be higher than t_i's difficulty rating. Prove: If we start at the right landmark, then we can hike at least $2m/n$ trails.

Problem 8 (1-10-100 problem, Alexander Soifer)**.** Given 100 many 10-elements sets such that any two sets have precisely one element in common, prove that there is one element that is common in all the sets.

Problem 9 (Passengers in a plane)**.** Passengers $P_1, \ldots, P_n$ enter a plane with n seats. Each passenger has a different assigned seat. The first passenger sits in the wrong seat. Thereafter, each passenger either sits in their seat if unoccupied or otherwise sits in a random unoccupied seat. What is the probability that the last passenger sits in his or her own seat?

Problem 10. (Spreading Infection) Some of the 64 cells of a chessboard are initially infected. Subsequently the infection spreads according to the following rule: if two neighbors of a cell are infected then the cell gets infected. No cell is ever cured. What is the minimum number of cells that need to be initially infected to guarantee that the infection spreads all over the chessboard? It is easy to see that 8 are sufficient in many ways. Prove that 7 are not enough.