Discrete Mathematics (Endsem Exam-2025)

May 1, 9:30 AM to 12:45 PM
Total: 100 points

1. Given an undirected simple graph G with n vertices and m edges, show that it must contain at least $m - n + 1$ cycles.

10 points

$4 < n$ $\frac{n(n-1)}{4}$ $n-1 < \binom{n}{2}$ $2n-2<n$ $n<2$

2. Prove the following.

(a) Take a complete graph and color each of its edges with red or green. Prove that for any possible coloring, there exists a spanning tree with all edges of the same color.

(b) Show that the number of spanning trees of the complete bipartite graph $K_{2,n}$ is $n \cdot 2^{n-1}$. A direct argument is possible (avoiding the matrix-tree theorem).

5+5=10 points

3. The following strategy is often used to prove that the number of some combinatorial objects is even: construct a graph, in which the objects correspond to vertices of odd degree. Use this strategy to show the following.

(a) If the graph G has even number of vertices and all of them have even degree, then it has even number of spanning trees.

(b) Let $G = (V, E)$ be a finite graph. Call a subset $A \subset V$ dominating if any vertex $v \in V - A$ has at least one neighbor in A. Then any graph has an odd number of dominating sets.

5+5=10 points

4. Use Hall's marriage theorem for bipartite matching to prove the following.

(a) (Existence of a system of distinct representatives/traversal) Let $(T_1, \ldots, T_n)$ be a family of subsets of $\{1, \ldots, n\}$. Suppose that there is a positive integer r such that each set T_i has cardinality r, and each point $j \in \{1, \ldots, n\}$ lies in exactly r of these sets. Then the family has an SDR. A system of distinct representatives or SDR for the $(T_1, \ldots, T_n)$ is an n-tuple $(a_1, \ldots, a_n)$ of elements of S such that $a_i \in T_i$ for all i, and $a_i \neq a_j$ for $i \neq j$.

(b) An $m \times n$ matrix consisting of the numbers $1, \ldots, n$ is called a *latin rectangle* if the same number never appears twice in any row or any column. Let $m < n$. Any $m \times n$ latin rectangle can be extended to an $(m+1) \times n$ latin rectangle by suitably adding a row. You may use the previous question on SDR.

4+6=10 points

Spanning

5. Let T and T' be two distinct trees on the same vertex set. Let e be an edge that is in T but not T'. Show that there exists an edge e' that is in T' but not T such that $T' + e - e'$ (adding e to T' and removing e') is also a tree. Use this to prove that for a weighted connected graph with

Spanning

$n^2 - 5n + 8 \le 0$ $\binom{n}{2} < 2n$ $n < 4$ $\binom{n-1}{2} + 1$

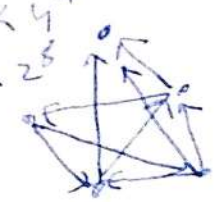
$\frac{(n-1)(n-2)}{2} + 1 \le \frac{n(n-1)}{4}$

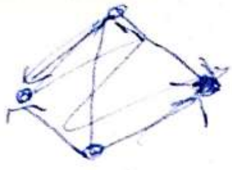
$2(n-1)(n-2) + 4 \le n(n-1)$

$3^5 \times 5 < 4^4$

$< 2^8$

$236 \quad \frac{6 \times 5}{2} \left(\frac{3}{4}\right)^n$





distinct edge weights, there is a unique minimal spanning tree. **10 points**

6. Recall that a tournament is an orientation of the complete graph. We call the vertices "players" and the arrow $a \to b$ indicates that player a beat player b. We say that player x dominates the set A of players if $(\forall a \in A)(x \to a)$. We say that the tournament is k-paradoxical if every set of k players is dominated by some player. Construct a 2-paradoxical tournament. Use the probabilistic method to prove that if $\binom{n}{k}(1-2^{-k})^{n-k} < 1$, then there exists a k-paradoxical tournament of n vertices.

$\binom{n}{k}(1-2^{-k})$

4+6=10 points

7. Let G be a graph with m edges, and let k be a positive integer. Prove that the vertices of G can be colored with k colors in such a way that there are at most m/k monochromatic edges (i.e., edges with both endpoints colored the same). Hint: linearity of expectation, probabilistic method. Let $0 < p < 1$ be the probability of getting head by tossing a biased coin. Let $P(n)$ be the probability of having an even number of heads after n coin tosses. Show that $P(n) = 1/2 + 1/2(1-2p)^n$. Hint: use the generating function for the number of heads. **5+5=10 points**

8. In the class, we discussed a greedy coloring argument showing that every graph G can be colored by $\Delta(G)+1$ colors where maximum degree of any vertex is $\Delta(G)$. Prove the following. Let G be a graph of maximum degree at most Δ. If every connected component of G contains a vertex of degree less than Δ, then the chromatic number $\chi(G) \leq \Delta$. Thus, if the graph is not regular, we can save a color. Give a proof using induction. **10 points**

9. Consider the simple undirected path on n vertices consisting of $n-1$ edges $\{i, i+1\}, 1 \leq i \leq n$. Suppose each vertex can be colored with any of k colors. What is the total number of colored paths we can get? However, the path can be flipped by applying the permutation $i \mapsto n-i+1$ without changing the adjacency relation. What is the number of distinct k-colorings of the path? Here two colorings are considered the same if we can reach one from the other by flipping the path. You can use the orbit counting lemma done in class. **10 points**

10. Let the cyclic group of prime order p, generated by g, act on the set of all p-tuples $(a_1, a_2, \ldots, a_p)$, $1 \leq a_i \leq n$ by right rotation. That is, g maps $(a_1, a_2, \ldots, a_p)$ to $(a_p, a_1, \ldots, a_{p-1})$. By examining the orbits of this group action prove that $n^p \equiv n \pmod p$. (note: no credit for any other proof) **10 points**

11. **(Optional problem)** Given a planar graph G, let V denote the number of vertices of G, E the number of edges of G, and F the number of faces of G (i.e. the number of two dimensional pieces G partitions the plane into). Euler's formula says for a connected planar graph $V - E + F = 2$. Prove Euler's formula by first proving the following statement: Euler's formula $V - E + F = 2$ holds for any planar graph that has an Eulerian tour (A graph has an Eulerian tour if and only if it's connected and every vertex has even degree). Show how to reduce the general planar graph case to planar Eulerian graph case by constructing a Eulerian graph from the planar graph. **0 points**

$n^p =$

$$\frac{\frac{n(n-1)}{2}\left(\frac{3}{4}\right)^{n-2}}{\frac{(n-1)(n-2)}{2}\left(\frac{3}{4}\right)^{n-3}} \qquad \frac{n}{n-2}\,\frac{3}{4}$$

2

$\binom{n}{k}\left(1-\left(\frac{1}{2}\right)^k\right)^{n-k} < 1$

$\binom{n}{2}\left(\frac{3}{4}\right)^{n-2}$

$n(n-1)$

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