Discrete Mathematics Extra Problem Sheet 3

1. Show that an undirected graph is 2 edge-connected if and only if we can orient /direct the edges such that the graph becomes strongly connected (there is a directed path between any two vertices).

2. Prove that the complete graph with 5 vertices is non-planar.

3. A tournament is a complete directed graph. A Hamiltonian path is a path that visits each vertex exactly once. Show that every tournament contains a Hamiltonian path.

4. Let $G = (V, E)$ be a simple graph. Suppose $\deg(v) \geq k$, $\forall v \in V$. Then G has cycles of at least $k - 1$ different lengths.

5. An *acyclic orientation* of a simple undirected graph is a way to assign directions to its edges such that the resulting digraph does not have any directed cycles. For example, it is easy to see that a forest with e edges has 2^e acyclic orientations. Show that the complete bipartite graph $K_{2,n}$ has $2 \cdot 3^n - 2^n$ acyclic orientations.

6. Prove that the vertex-connectivity of a graph is less than or equal to its edge-connectivity.

7. Show that a map formed on a plane by finitely many circles is 2-colourable.

8. Take a graph $G = (V, E)$, and let $\mathcal{I} = \{F \subseteq E : F \text{ is a matching}\}$. Show that $(E, \mathcal{I})$ does not define a matroid.

9. The chromatic number of an undirected graph G is equal to the minimum number of vertices, over all orientations of G, in the longest oriented path in G. (An orientation of an undirected graph G is a directed graph obtained by giving each edge in G a direction.)

10. Let G be a connected graph with an even number of vertices. Prove that you can select a subset of edges of G such that each vertex is incident to an odd number of selected edges.