Discrete Mathematics (Midsem Exam)

March 6, 2025, 9.30 to 11:30
Total: 100 points

Answer all questions.

1. Let A and B be infinite sets and $f : A \to B$ a surjection such that $f^{-1}(b)$ is either finite or countable for all $b \in B$. Show that $|A| = |B|$. **10 points** (2023 qdy)

2. Suppose A is an uncountable subset of reals. Show that there is a real number a such that the subsets $A \cap (-\infty, a)$ and $A \cap (a, \infty)$ are both uncountable. **10 points**

3. Suppose in an election between two candidates A and B, A gets a votes and B gets b votes, with $a > b$. If the votes are counted in random order, what is the probability that A always stays ahead of B? **10 points**

4. Using generating functions, count the number of solutions s_n to $a + b + c = n$ in nonnegative integers a, b, c such that a is a multiple of 3, $b \leq 2$, and $c \geq 1$. Find a closed form for the generating function of the sequence $(s_n)_{n \geq 0}$. **10 points**

5. For a tree T with vertex set $[n]$, its Prüfer code is a sequence $(a_1, a_2, \ldots, a_{n-2}), a_i \in [n]$ formed by repeatedly deleting the least labeled leaf in the remaining tree and recording its neighbor. Thus, a_i is the label of the neighbor of the least labeled leaf in the tree at the i^{th} stage. Show that the Prüfer code gives a bijection from the set of labeled trees on $[n]$ to the set of sequences $[n]^{n-2}$. **10 points**

6. Let $n = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$ be the prime factorization of n. Show that the poset $D(n)$ of divisors of n is isomorphic to the direct product poset $D(p_1^{e_1}) \times D(p_2^{e_2}) \times \cdots \times D(p_k^{e_k})$. Using this derive the Möbius function of $D(n)$. **10 points**

7. Let $A = (a_1, a_2, \ldots, a_n)$ and $B = (b_1, b_2, \ldots, b_n)$ be two sequences consisting of $2n$ distinct integers such that $a_i < b_i$ for each i. Suppose $(a_1', a_2', \ldots, a_n')$ and $(b_1', b_2', \ldots, b_n')$ are the sequences A and B sorted in decreasing order, respectively. Show that $a_i' < b_i'$ for all i. **10 points**

8. Let X be a finite universe and $A_1, A_2, \ldots, A_n \subseteq X$. Suppose for every subset of indices $J \subseteq [n]$ we have $|\cup_{i \in J} A_i| \geq |J|$. Show using Dilworth's theorem on finite posets that we can find n distinct elements $x_1, x_2, \ldots, x_n \in X$ such that $x_i \in A_i, i \in [n]$. **10 points**

9. Given positive integers m, n, p show, using Ramsey's theorem, that there is a function $f(m, n, p)$ such that in any sequence $a_1, a_2, \ldots, a_N$ of real numbers of length $N \geq f(m, n, p)$ either there is a strictly increasing subsequence of length m or strictly decreasing subsequence of length n or a subsequence of length p with all equal elements. **10 points**

10. Let X be a finite universe and $f, g : 2^X \to \mathbb{R}$. Show that the following are equivalent:

- $g(I) = \sum_{J \subseteq I} f(J)$.

$f(J) = 1$ $\boxed{mnp}$ $mn+1$ distinct

$mnp+1$ $mn+1$ mnp seq

mnp

$a_1 \quad a_2 \quad a_3 \ldots$

- $f(I) = \sum_{J \subseteq I} (-1)^{|I \setminus J|} g(J)$.

Deduce that the $(n+1) \times (n+1)$ matrices A and B defined by $A_{ij} = \binom{i}{j}$ and $B_{ij} = (-1)^{i+j}\binom{i}{j}$ are inverses of each other. **10 points**

$g(I) = j \quad 1 - 4 + 6 - 4 + 1$

$(-1)^{i-j}$

$(-1)^{(j)} \binom{j}{k}$

$1\ 2\ 0\ 2\ 0\ 0 \ldots 0$

$\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$

$f(i, i) = 1$

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$\underbrace{\sum_{k=1}^{n+1} \binom{i}{k} \cdot \binom{k}{j} (-1)^{k+j}}_{i}$$

$\dfrac{i!}{(k!)(i-k!)} \quad \dfrac{k!}{j!\,(j-k)!} \quad k-j$

$i > k > j$.

$\dfrac{i!}{0!} \quad (1+x)^k$

$x^{j}(1-x)^k$

$(1+x)^i$

x^{k+j}