Discrete Mathematics Problem Sheet 1

Star marked exercises could be harder.

1. Consider the power set of $\mathbb{N}$ as $\{0,1\}^{\mathbb{N}}$. A partial order for it is defined as follows. given two elements $\mathbf{a}, \mathbf{b} \in \{0,1\}^{\mathbb{N}}$, that is, sequences $\mathbf{a} = (a_1, a_2, \dots)$ and $\mathbf{b} = (b_1, b_2, \dots)$, define $\mathbf{a} \leq \mathbf{b}$ if $a_i \leq b_i$ for all $i \in \mathbb{N}$. An antichain is a subset of a partially ordered set such that any two distinct elements in the subset are incomparable.

   (a) Give a countably infinite chain in $\{0,1\}^{\mathbb{N}}$..

   (b) Find a countably infinite antichain in $\{0,1\}^{\mathbb{N}}$.

   (c) Find an uncountable antichain in $\{0,1\}^{\mathbb{N}}$.

   (d) (*+) Find an uncountable chain in $\{0,1\}^{\mathbb{N}}$.

   (e) (*-) Find an uncountable collection of subsets of $\mathbb{N}$ such that any two subsets have finite intersection.

2. Are the sets below countable or uncountable? Justify.

   (a) All non-increasing functions from $\mathbb{N}$ to $\mathbb{N}$. Non-increasing function f means $f(i) \geq f(i+1)$ for all $i \geq 1$.

   (b) All non-decreasing functions from $\mathbb{N}$ to $\mathbb{N}$. That is, functions f such that $f(i) \leq f(i+1)$ for all i.

   (c) All injective functions from $\mathbb{N}$ to $\mathbb{N}$.

   (d) All surjections from $\mathbb{N}$ to $\mathbb{N}$.

   (e) All bijections $\mathbb{N}$ to $\mathbb{N}$.

3. An infinite binary sequence is *lonely* if there are no consecutive 1's in it. Show that the set of lonely sequences is uncountable.

4. (The Hausdorff maximal principle). Let S be a partially ordered set. Then S contains a maximal chain (i.e. a chain which is not contained in a bigger chain). Use Zorn's lemma to prove this.

5. (Ultrafilter lemma) A *filter* on a set X is a set of subsets $\mathcal{F} \subseteq 2^X$ satisfying

(a) $X \in \mathcal{F}$ (intuitively, the whole set is *large*).

(b) $\varnothing \notin \mathcal{F}$ (the empty set is not large).

(c) If $A \in \mathcal{F}$ and $B \in \mathcal{F}$ then $A \cap B \in \mathcal{F}$ (large sets have large intersection).

(d) If $A \in \mathcal{F}$ and $B \supseteq A$ then $B \in \mathcal{F}$ (any set containing a large set is large).

A filter $\mathcal{F}$ on X is an *ultrafilter* if for any $A \subseteq X$, $A \in \mathcal{F}$ or $X \setminus A \in \mathcal{F}$ (every set is either large or co-large). Using Zorn's lemma, prove that every filter is contained in an ultrafilter.

6. Let X be an infinite set, and $\mathcal{F}$ a family of *finite subsets* of X. A subfamily $\mathcal{F}'$ of $\mathcal{F}$ is called a Δ-system if there is a fixed subset $S \subset X$ such that $A \cap B = S$ for all distinct $A, B \in \mathcal{F}'$.

   - Suppose all sets in the family $\mathcal{F}$ are of size n, for a natural number n. Show by induction on n that $\mathcal{F}$ contains an infinite Δ-system.
   - Suppose the family $\mathcal{F}$ is uncountable. Show that $\mathcal{F}$ contains an uncountable Δ-system.

7. Let $n = 2k$, and X a set of n elements. Define a *factor* to be a partition of X into k sets of size 2. Count the number of factors of X.

8. Give a bijective proof for $\sum_{k=0}^{n} k\binom{n}{k} = n2^{n-1}$. Also give a computational proof.

9. Let $0 \leq \ell \leq k \leq n$. Show that $\binom{n}{k}\binom{k}{\ell} = \binom{n}{\ell}\binom{n-\ell}{k-\ell}$ with a bijective proof.

10. A composition of n is a sequence $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ of positive integers such that $\sum_i \alpha_i = n$. The number of compositions of n is 2^{n-1}. Give a bijective argument.

11. (*-) If exactly k summands appear in a composition α, then we say that α has k parts. Show that the total number of parts of all compositions of n is equal to $(n+1)2^{n-2}$.

12. (*-) For $n \geq 2$, show that the number of compositions of n with an even number of even parts is equal to 2^{n-2}.