Discrete Mathematics Problem Sheet 2

Feb 18, 2025

1. Find a closed form for the generating function g(x) of the sequence $a_n = n^2$, $n \geq 1$. Obtain from it the generating function for the sequence $b_n = \sum_{i=1}^{n} i^2$ and hence a closed form for $\sum_{i=1}^{n} i^2$.

2. Suppose that in your wallet you have 3 twenty-dollar bills, 5 tens, and 6 fives. Find a formula for the polynomial $P(x)$ in which the coefficient of x^n is the number of ways you can change for n dollars, under each of the following two assumptions: (a) Two change combinations are considered the same if they contain the same number of bills of each denomination, i.e., we treat the bills of a given denomination as indistinguishable. (b) We count change combinations as different if the individual bills used are different, i.e., we treat all individual bills as distinguishable (for instance, by looking at their serial numbers).

3. Let q_n be the number of ways of partitioning a set S of size n into 2 cycles, one colored red and the other blue (so we'll count as different two permutations which are identical, but whose cycles are colored differently). Come up with the exponential generating function corresponding to the sequence (q_n).

4. Let $p_{odd}(n)$ be the number of partitions of the integer n into odd parts, with $p_{odd}(0) = 1$. Find the generating function of $(p_{odd}(n))$. Let $p_d(n)$ be the number of partions of integer n into distinct parts, with $p_d(0) = 1$. Find the generating function of the numbers $p_d(n)$.

5. Among $n + 2$ arbitrary integers show that either there are two whose difference is divisible by $2n$ or there are two whose sum is divisible by $2n$.

6. Let a_n be the number of ways of drawing n balls from an unlimited supply of red, blue, and green balls such that the number of green balls drawn is always even. Give a closed form for the generating function of the sequence a_n, $n \geq 0$.

7. Determine the sum $\sum_{i=1}^{n} i \cdot \binom{n}{i}$ by using generating functions. Is there a combinatorial argument for it as well?

8. . What is the generating function closed form for $\binom{2n}{n}$.

9. How many r-digit binary sequences are there with no adjacent zeros?

10. Solve $a_r - a_{r-1} = r!$, $r \geq 1$ with $a_0 = 2$.

11. Suppose in an election between two candidates A and B, A gets a votes and B gets b votes, with $a > b$. If the votes are counted in random order, what is the probability that A always stays ahead of B?

12. Consider $2n$ points placed around a circle. How many ways are there (as a function of n) of pairing them up with n nonintersecting chords?

13. Consider the set S of length-$2n$ binary sequences with equal number of 0's and 1's such that every prefix has at least as many 1's as 0's. For $w \in S$, let $\bar{w}$ denote its bitwise complement and w^R denote its reverse. Let $T = \{w \in S \mid w = \bar{w}^R\}$. What is $|T|$ as a function of n?

14. For a tree T with vertex set $[n]$, its Prüfer code is formed by repeatedly deleting the least labeled leaf in the remaining tree and recording the name of its neighbor $n-2$ times. Prove that the Prüfer code gives a bijection from the set of labeled trees on $[n]$ to the set of sequences $[n]^{n-2}$.

15. What will be the Prüfer code of a tree T on $[n]$ such that the degree of i is d_i for $1 \leq i \leq n$? Show that the number of such labeled trees is the multinomial cofficient $\binom{n-2}{d_1-1\ d_2-1\ \cdots\ d_n-1}$.