Discrete Mathematics Problem Sheet 3

Feb 28, 2025

1. Let X be a finite universe and $A_i \subseteq X, 1 \leq i \leq n$. Using the version of inclusion-exclusion principle derive the expression for $| \cup_{i=1}^{n} A_i|$ in terms of $|A_I|, I \subseteq [n]$.

2. Let G be a graph on n vertices with edge set E. A proper k-coloring of G is a map $c : [n] \to [k]$ such that for all $\{u, v\} \in E$ we have $c(u) \neq c(v)$. Using inclusion-exclusion principle for counting the number of proper k-colorings, show that
$$\sum_{S \subseteq E} (1)^k k^{c(S)},$$
where $c(S)$ is the number of *connected components* of the graph $G_S = ([n], S)$. A connected component of G_S is a maximal subset of vertices U of $[n]$ such that any two vertices in U are reachable from each other using only edges in S.

3. Let P be a finite poset and μ its Möbius function. For any two elements $a, b \in P$ show that $\mu(a, b) = \sum_{i \geq 0} (-1)^i c_i$, where c_i is the number of chains in P between a and b.

4. For a prime p, let $\mathbb{F}_p$ denote the finite field with p elements in it (field operations are addition and multiplication modulo p). Let $(P, \subseteq)$ be the poset of all subspaces of the n-dimensional vector space $\mathbb{F}_p^n$, partially ordered by containment. What is $|P|$? Determine the Möbius function of P.

5. How many permutations π are there in S_n such that $\pi(i) \neq i$ for every even i.

6. Suppose there are $2n$ persons, each holding a number from $[n]$ such that for each $i \in [n]$ exactly two persons are holding i. If we randomly line up these $2n$ persons, what is the probability that no two consecutive persons hold the same number?

7. Determine the number of graphs on n vertices with exactly m edges and exactly k isolated vertices (an isolated vertex is one which has no edges incident on it).

8. For a permutation $\pi \in S_n$, an index $i \in [n]$ is a *descent* if $\pi(i) > \pi(i+1)$. The set $D(\pi) = \{i < n \mid \pi(i) > \pi(i+1)\}$ is the descent set of π. For a subset S of $[n-1]$ count the number of permutations in S_n whose descent set is contained in S. Using that count the number of permutations whose descent set if exactly S.