$\alpha + \alpha = 2\alpha$ $\quad \alpha = \beta \cdot \beta$

$\alpha \leq 2\alpha$ $\quad \boxed{\alpha + \beta = \alpha}$

$\alpha < \alpha + \beta < \alpha$ $\quad 2\alpha \geq \alpha \quad \alpha. \quad \alpha + \beta \geq \alpha$

$\alpha \geq \alpha + \beta$

Discrete Mathematics Quiz 1

$\alpha \cdot \beta \leq \alpha$ $\quad \alpha + \alpha \quad (\alpha_1, \alpha_2 \ldots \alpha_n)$

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$\alpha \leq \alpha \cdot \beta \leq \alpha$ $\quad \alpha \leq 2\alpha$

Write clear and concise solutions. All problems carry equal marks.

1. For any three infinite cardinal numbers α, β and γ show that $\alpha \cdot (\beta + \gamma) \in \{\alpha, \beta, \gamma\}$.

   Proof of Lone?

2. Suppose $G = (V, E)$ is an infinite graph. For every $v \in V$ its neighborhood in G is defined as $N(v) = \{u \in V \mid \{u, v\} \in E\}$.

   (a) Suppose $N(v)$ is finite for each $v \in V$. Show that G can be properly colored with countably many colors. I.e. show that there is a coloring $c : V \to \mathbb{N}$ such that for all $\{u, v\} \in E$, $c(u) \neq c(v)$. Give an example of such a graph which cannot be properly colored with finitely many colors.

   (b) Suppose $|N(v)| \leq \alpha$ for each $v \in V$, where α is an infinite cardinal number. Show that G can properly colored with a color set of size α.

3. A *pairing* of the elements of $[2k] = \{1, 2, \ldots, 2k\}$ is a partition of $[2k]$ into sets of size 2 $\{i_1, j_1, \}, \{i_2, j_2\}, \ldots, \{i_k, j_k\}$. How many distinct pairings does $[2k]$ have? Justify with proof.

4. Let p be a prime. Show that $(1 + x)^p = (1 + x^p) (\text{mod } p)$. Show, as a consequence, that $\binom{pm}{p\ell} = \binom{m}{\ell} (\text{mod } p)$. Furthermore, if we write $a = \sum_{i=0}^{k} a_i p^i$ and $b = \sum_{i=0}^{k} b_i p^i$, where $0 \leq a_i, b_i \leq p - 1$ for all i then $\binom{a}{b} = \prod_{i=0}^{k} \binom{a_i}{b_i} (\text{mod } p)$.

$\dfrac{(pm)!}{(p\ell)!\,(pm-\ell)!}$

$(1+x)^{a_0} \quad (1+x)^{a_1 p} \quad (1+x)^{a_2 p^2} + \cdots \quad p^m$

$p^\ell \quad p^{m\ell}$ $\quad \alpha_i k \quad \alpha_i k.$

$a_1 \leftarrow a_1$

$a_2 \leftarrow a_2$

$\vdots \quad \vdots$

$a_n \leftarrow a_n$

β_1

β_2

$\vdots$

β_n

$\alpha k \geq \alpha$

$\boxed{\alpha \geq \alpha k}$

$\alpha + \beta < \alpha + \alpha = 2\alpha = \alpha$

$\alpha + \beta < \alpha$

$\underbrace{\alpha + \alpha + \ldots + \alpha}$