# Discrete Mathematics (Quiz 2)

**Date and Time: April 15, 2025**
**6 p.m to 7 p.m**

1. Prove that if all the vertices of a bipartite graph $G$ have the same degree $d$, then it has a perfect matching. Let $E_o$ be a set of any $d-1$ edges in $G$. Show that $G - E_o$ has a perfect matching. **5+5=10 points**

2. There are $n$ people who need to be assigned to $n$ houses. Each person ranks the houses in some order, with no ties. After the assignment is made, it is observed that every other assignment would assign at least one person to a house that the person ranks lower than the house in the given assignment. Prove that at least one person receives his or her top choice in the given assignment.
**10 points**

3. Let $G$ be a simple graph such that the minimum degree of any vertex is $\geq k$. If $k \geq 2$, show that there is a cycle in the graph of length at least $k+1$. **10 points**

4. Let us consider a complete directed graph/tournament (between every pair of vertices, there exists exactly one directed edge). A *king* in a tournament is a vertex $v$ such that every other vertex is reachable from $v$ via a directed path of length at most 2. Prove that in any tournament there is at least one king. Can there be more than one king in a tournament? **10 points**

5. Let $G$ be an undirected connected graph with distinct edge weights. Let $e_{max}$ be the edge with maximum weight and $e_{min}$ the edge with minimum weight. What can you say about the following statements? Give short justifications supporting your claims.

   I Every minimum spanning tree of $G$ must contain $e_{min}$

   II Every minimum spanning tree must exclude $e_{max}$

   III Every maximum spanning tree must include $e_{max}$.

**10 points**