Probability Theory, HW 2 Solution Notes

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This is a compilation of solutions for Assignment 2 of Probability Theory class instructed by Prof. R. Srinivasan. The solutions are of original design. The template is a modified template of the one used for USAMO solutions by Evan Chen.

Contents

§0 Problems

1. If X is a continuous random variable with distribution function F and density function f, show that the random variable $Y = |X|$ is also continuous and express (with proof) its cumulative distribution function and density in terms of F and f. Find the density of Y when X has (i) normal distribution (ii) exponential distribution and (iii) the Cauchy distribution.

2. Show that the $\int_{-\infty}^{\infty} |x\mu| f(x) \mathrm{d}x$ becomes minimum when μ is the median of the distribution with density f. (For continuous distribution median is the point $x_0 \in \mathbb{R}$ such that $F(x_0) = \frac{1}{2}$.)

3. Show that the function

$$F(x, y) = \begin{cases} 0, & \text{if } x + y < 1 \\ 1, & \text{if } x + y \geq 1 \end{cases}$$

is not a joint distribution function.

4. If the $\log X$ is normally distributed find the density X.

5. Let f be the density function of the random variable X. Suppose that X has a symmetric distribution about a. Show that the mean $\mathbb{E}(X)$ equals a, provided it exists

6. If $\mathbb{E}(X) = \mathbb{E}(X^2) = 0$, show that $\mathbb{P}(X = 0) = 1$

7. Let X and Y have the joint density

$$f(x, y) = cx^{n_1 - 1}(y - x)^{n_2 - 1} e^{-y}$$

with $0 < x < y < \infty$.
Find (a) the constant c, (b) the marginal distributions of X and Y.

8. Calculate the characteristic function of a Gamma distribution with parameters λ and α and deduce the characteristic function of χ^2

9. Let X_1 and X_2 be independent exponential variables, parameter λ. Find the joint density function of (Y_1, Y_2) where $Y_1 = X_1 + X_2$, $Y_2 = \frac{X_1}{X_2}$, and show that they are independent.

§1 Solutions

§1.1 Problem 1

Problem statement

If X is a continuous random variable with distribution function F and density function f, show that the random variable $Y = |X|$ is also continuous and express (with proof) its cumulative distribution function and density in terms of F and f. Find the density of Y when X has (i) normal distribution (ii) exponential distribution and (iii) the Cauchy distribution.

Proof. For $k < 0$, $K_Y(k) = 0$ as $Y = |X| \geq 0$. For $k \geq 0$,

$$\begin{aligned} F_Y(k) &= \mathbb{P}(-k \leq X \leq k) \\ &= F_X(k) - F_X(-k) \end{aligned}$$

As X is a continuous random variable, the CDF is differentiable except at a measure zero set. Thus, differentiating on both sides

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}k} F_Y(k) &= \frac{\mathrm{d}}{\mathrm{d}k}(F_X(k) - F_X(-k)) \\ \implies f_Y(k) &= f_X(k) + f_X(-K) \end{aligned}$$

For $k < 0$, $f_Y(k) = 0$ as $Y = |X| \geq 0$ as PDF is 0 for $Y < 0$. (i)

$$\begin{aligned} f_Y(k) =& f_X(-k) + f_X(k) \\ =& \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(-k-\mu)^2}{2\sigma^2}\right) + \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(k-\mu)^2}{2\sigma^2}\right) \\ =& \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(k+\mu)^2}{2\sigma^2}\right) + \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(k-\mu)^2}{2\sigma^2}\right) \\ =& \frac{1}{\sqrt{2\pi}\sigma} \left(\exp\left(-\frac{k^2+2k\mu+\mu^2}{2\sigma^2}\right) + \exp\left(-\frac{k^2-2k\mu+\mu^2}{2\sigma^2}\right)\right) \\ =& \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{k^2+\mu^2}{2\sigma^2}\right)\left(\exp\left(\frac{-2k\mu}{2\sigma^2}\right) + \exp\left(\frac{2k\mu}{2\sigma^2}\right)\right) \end{aligned}$$

(ii)

$$\begin{aligned} f_Y(k) =& f_X(-k) + f_X(k) \\ &= 0 + \alpha e^{-\alpha k} \\ &= \alpha e^{-\alpha k} \end{aligned}$$

(iii)

$$\begin{aligned} f_Y(k) =& f_X(-k) + f_X(k) \\ &= \frac{1}{\pi}\left(\frac{\gamma}{(-k-x_0)^2+\gamma^2}\right) + \frac{1}{\pi}\left(\frac{\gamma}{(k-x_0)^2+\gamma^2}\right) \\ &= \frac{\gamma}{\pi}\left(\frac{1}{(k+x_0)^2+\gamma^2} + \frac{1}{(k-x_0)^2+\gamma^2}\right) \end{aligned}$$

□

§1.2 Problem 2

Problem statement

Show that the $\int_{-\infty}^{\infty} |x-\mu| f(x)\mathrm{d}x$ becomes minimum when μ is the median of the distribution with density f. (For continuous distribution median is the point $x_0 \in \mathbb{R}$ such that $F(x_0) = \frac{1}{2}$.)

Proof. Define $g(\mu) := \int_{-\infty}^{\infty} |x-\mu| f(x)\mathrm{d}x$ then,

$$\begin{aligned} g'(\mu) =& \frac{\mathrm{d}}{\mathrm{d}\mu} \int_{-\infty}^{\infty} |x-\mu| f(x)\mathrm{d}x \\ =& \int_{-\infty}^{\infty} \operatorname{sgn}(x-\mu) f(x)\mathrm{d}x \\ =& -\int_{-\infty}^{\mu} f(x)\mathrm{d}x + \int_{\mu}^{\infty} f(x)\mathrm{d}x \\ =& -F(\mu) + (1-F(\mu)) \\ =& 1 - 2F(\mu) \end{aligned}$$

Setting $g'(\mu) = 0 \implies F(\mu) = 1/2 => \mu = F^{-1}(0.5)$ that is μ is median, say M. To prove this is a maxima, the first derivative test suffices as for $\mu < M$, $1 - 2F(\mu) > 0$ as F is increasing and otherwise $\mu > M$ then $1 - 2F(\mu) < 0$ is decreasing making $\mu = M$ the maxima.

□

§1.3 Problem 3

Problem statement

Show that the function

$$F(x,y) = \begin{cases} 0, & \text{if } x+y < 1 \\ 1, & \text{if } x+y \geq 1 \end{cases}$$

is not a joint distribution function.

Proof. Proof by contradiction. Let F define a joint distribution function. Notice

$$\mathbb{P}(a \leq X \leq b, c \leq Y \leq d) = F(b,d) - F(a,d) - F(b,c) + F(a,c)$$

As probability is always non-negative, $\mathbb{P}(0 \leq X \leq 2, 0 \leq Y \leq 2) \geq 0$ but

$$F(2,2) - F(2,0) - F(0,2) + F(0,0) = 1 - 1 - 1 + 0 = -1$$

As $-1 < 0$, we have a contradiction.
Thus, F is not a valid joint distribution function. □

§1.4 Problem 4

Problem statement

If the $\log X$ is normally distributed find the density X.

Proof. Let $\mathrm{Nor}(\mu, \sigma^2) \sim Y = \log X \implies X = e^(Y)$ where Y is normally distributed. This implies

$$\begin{aligned} &\mathbb{P}(X \leq x) = \mathbb{P}(Y \leq \log(x)) \\ \implies &f_X(x) = f_Y(\log(x))\frac{1}{x} \\ \implies &f_X(x) = \frac{1}{x\sqrt{2\pi\sigma}} \exp\left(-\frac{(\log(x) - \mu)^2}{2\sigma^2}\right) \end{aligned}$$

□

§1.5 Problem 5

Problem statement

Let f be the density function of the random variable X. Suppose that X has a symmetric distribution about a. Show that the mean $\mathbb{E}(X)$ equals a, provided it exists

Proof. As the distribution is symmetric about a, $f(a + x) = f(a - x)$ for all $x \in \mathbb{R}$. Making the substitution $x = a + k$

$$\begin{aligned} \mathbb{E}(X) &= \int_{-\infty}^{\infty} xf(x)\mathrm{d}x \\ &= \int_{-\infty}^{\infty} (a + k)f(a + k)\mathrm{d}k \\ &= a\int_{-\infty}^{\infty} f(a + k)\mathrm{d}k + \int_{-\infty}^{\infty} kf(a + k)\mathrm{d}k \\ &= a + \int_{-\infty}^{0} kf(a + k)\mathrm{d}k + \int_{0}^{\infty} kf(a + k)\mathrm{d}k \end{aligned}$$

We will now make the substitution $k' = -k \implies \mathrm{d}k' = -\mathrm{d}k$

$$\begin{aligned} &= a - \inf_0^{-\infty} -kf(a + k)\mathrm{d}k' + \int_0^{\infty} kf(a + k)\mathrm{d}k \\ &= a + \inf_0^{\infty} k'f(a - k')\mathrm{d}k' + \int_0^{\infty} kf(a + k)\mathrm{d}k \\ &= a + \int_0^{\infty} kf(a + K) + k'f(a - k')\mathrm{d}k \\ &= a - \int_0^{\infty} (k - k)f(a + k)\mathrm{d}k \\ &= a \end{aligned}$$

Note, the combining of the integrals could only be done if both the integrals were finite as otherwise $\pm\infty\pm\infty$ is undefined. Hence, this computation is only valid if $\mathbb{E}(a+K) \implies \mathbb{E}(X)$ exists as $X = a + K$. □

§1.6 Problem 6

Problem statement

If $\mathbb{E}(X) = \mathbb{E}(X^2) = 0$, show that $\mathbb{P}(X = 0) = 1$

Proof. Notice, $\sigma^2(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = 0 - 0^2 = 0$. Thus, by Chebyshev,

$$\begin{aligned} \mathbb{P}(|X - \mu| \geq k) &\leq \frac{\sigma^2}{k^2} \\ \implies \mathbb{P}(|X - 0| \geq k) &\leq \frac{0}{k^2} \\ \implies \mathbb{P}(|X| \geq k) &\leq 0 \end{aligned}$$

As probability is non-negative, $\mathbb{P}(|X| \geq k) \leq 0 \implies \mathbb{P}(X \geq k) = 0 \implies \mathbb{P}(X \neq 0) = 0 \implies \mathbb{P}(x = 0) = 1 - 0 = 1$. □

§1.7 Problem 7

Problem statement

Let X and Y have the joint density

$$f(x, y) = cx^{n_1-1}(y - x)^{n_2-1}e^{-y}$$

with $0 < x < y < \infty$.
Find (a) the constant c, (b) the marginal distributions of X and Y.

Proof. (a) We will use the fact that $\int_0^\infty \int_x^\infty f(x, y)\mathrm{d}y\mathrm{d}x = 1$, given $0 < x < y < \infty$. Making the substitution $t = y - x \implies \mathrm{d}t = \mathrm{d}y$ and

$$\begin{aligned} 1 &= \int_0^\infty \int_x^\infty cx^{n_1-1}(y - x)^{n_2-1}e^{-y}\mathrm{d}y\mathrm{d}x \\ &= \int_0^\infty cx^{n_1-1} \int_0^\infty t^{n_2-1}e^{-t-x}\mathrm{d}t\mathrm{d}x \\ &= \int_0^\infty cx^{n_1-1}e^{-x} \int_0^\infty t^{n_2-1}e^{-t}\mathrm{d}t\mathrm{d}x \\ &= \int_0^\infty cx^{n_1-1}e^{-x}\Gamma(n_2)\mathrm{d}x \\ &= c\Gamma(n_2) \int_0^\infty x^{n_1-1}e^{-x}\mathrm{d}x \\ &= c\Gamma(n_1)\Gamma(n_2) \end{aligned}$$

$$\implies c = \frac{1}{\Gamma(n_1)\Gamma(n_2)}$$

(b) For the marginal density of Y, using $0 < x < y$

$$\begin{aligned}
f_Y(y) &= \int_{-\infty}^{\infty} f(x,y)\mathrm{d}x \\
&= \int_0^y \frac{1}{\Gamma(n_1)\Gamma(n_2)} x^{n_1-1}(y-x)^{n_2-1}e^{-y}\mathrm{d}x \\
&= \frac{1}{\Gamma(n_1)\Gamma(n_2)} e^{-y} \int_0^y x^{n_1-1}(y-x)^{n_2-1}\mathrm{d}x
\end{aligned}$$

Making the substitution $x = yt \implies \mathrm{d}x = y\mathrm{d}t$.

$$\begin{aligned}
f_Y(y) &= \frac{1}{\Gamma(n_1)\Gamma(n_2)} e^{-y} y \int_0^y (yt)^{n_1-1}(y-yt)^{n_2-1}\mathrm{d}t \\
&= \frac{1}{\Gamma(n_1)\Gamma(n_2)} e^{-y} y^{n_1+n_2-1} \int_0^1 t^{n_1-1}(1-t)^{n_2-1}\mathrm{d}t \\
&= \frac{1}{\Gamma(n_1)\Gamma(n_2)} e^{-y} y^{n_1+n_2-1} B(n_1,n_2) \\
&= \frac{y^{n_1+n_2-1}1^{n_1+n_2}}{\Gamma(n_1+n_2)} e^{-1y} \\
&= \mathrm{Gamma}(n_1+n_2,1)(y)
\end{aligned}$$

Thus, $Y \sim \mathrm{Gamma}(n_1+n_2,1)$ For the marginal of X, using $x < y < \infty$ and making the substitution $t = y - x \implies \mathrm{d}t = \mathrm{d}y$

$$\begin{aligned}
f_X(x) &= \int_{-\infty}^{\infty} f(x,y)\mathrm{d}y \\
&= \int_x^{\infty} \frac{1}{\Gamma(n_1)\Gamma(n_2)} x^{n_1-1}(y-x)^{n_2-1}e^{-y}\mathrm{d}y \\
&= \frac{1}{\Gamma(n_1)\Gamma(n_2)} x^{n_1-1} \int_x^{\infty} (y-x)^{n_2-1}e^{-y}\mathrm{d}y \\
&= \frac{1}{\Gamma(n_1)\Gamma(n_2)} x^{n_1-1} \int_0^{\infty} (t)^{n_2-1}e^{-t-x}\mathrm{d}t \\
&= \frac{1}{\Gamma(n_1)\Gamma(n_2)} x^{n_1-1}e^{-x} \int_0^{\infty} (t)^{n_2-1}e^{-t}\mathrm{d}t \\
&= \frac{1}{\Gamma(n_1)\Gamma(n_2)} x^{n_1-1}e^{-x}\Gamma(n_2) \\
&= \frac{x^{n_1-1}1^{n_1}}{\Gamma(n_1)} e^{-1x} \\
&= \mathrm{Gamma}(n_1,1)(x)
\end{aligned}$$

Thus, $X \sim \mathrm{Gamma}(n_1,1)$. □

§1.8 Problem 8

Problem statement

Calculate the characteristic function of a Gamma distribution with parameters λ and α and deduce the characteristic function of χ^2

Proof. Let $X \sim \text{Gamma}(\alpha, \lambda)$. Then,

$$\begin{aligned}
&\mathbb{E}(e^{itX}) \\
&= \int_0^\infty e^{itx} \frac{\lambda^\alpha x^{\alpha-1}}{\Gamma(\alpha)} e^{-\lambda x} \mathrm{d}x \\
&= \frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^\infty e^{itx} x^{\alpha-1} e^{-\lambda x} \mathrm{d}x \\
&= \frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^\infty x^{\alpha-1} e^{-(\lambda-it)x} \mathrm{d}x
\end{aligned}$$

Making the substitution, $(\lambda - it)x = s \implies (\lambda - it)\mathrm{d}x = \mathrm{d}s$

$$\begin{aligned}
\mathbb{E}(e^{itX}) &= \frac{\lambda^\alpha}{\Gamma(\alpha)(\lambda - it)} \int_0^\infty \left(\frac{s}{\lambda - it}\right)^{\alpha-1} e^{-s} \mathrm{d}s \\
&= \frac{\lambda^\alpha}{\Gamma(\alpha)(\lambda - it)^\alpha} \int_0^\infty s^{\alpha-1} e^{-s} \mathrm{d}s \\
&= \frac{\lambda^\alpha}{\Gamma(\alpha)(\lambda - it)^\alpha} \Gamma(\alpha) \\
&= \frac{\lambda^\alpha}{(\lambda - it)^\alpha} \\
&= \left(\frac{\lambda}{\lambda - it}\right)^\alpha \\
&= \left(\frac{1}{1 - \frac{it}{\lambda}}\right)^\alpha \\
&= \left(1 - \frac{it}{\lambda}\right)^{-\alpha}
\end{aligned}$$

As $\chi_k^2 \sim \text{Gamma}(\frac{k}{2}, \frac{1}{2})$, therefore it's characteristic function is same as that for $\text{Gamma}(\frac{k}{2}, \frac{1}{2})$ by the uniqueness of characteristic functions.
Thus, the characteristic function of χ_k^2 is $(1 - 2it)^{-\frac{k}{2}}$. □

§1.9 Problem 9

Problem statement

Let X_1 and X_2 be independent exponential variables, parameter λ. Find the joint density function of (Y_1, Y_2) where $Y_1 = X_1 + X_2$, $Y_2 = \frac{X_1}{X_2}$, and show that they are independent.

Proof. By the definition of Gamma distribution as sum of waiting times of exponential distribution, $Y_1 \sim \text{Gamma}(\lambda, 2) \implies f_{Y_1}(y_1) = \lambda^2 y_1 e^{-\lambda y_1}$.

Now consider

$$F_{Y_2}(y_2) = \int_0^\infty F_{X_2}(y_2 x) f_{X_1}(x) \mathrm{d}x$$

$$
\begin{aligned}
&= \int_0^\infty (1 - e^{-\lambda y_2 x})\lambda e^{-\lambda x} \mathrm{d}x \\
&= \lambda \int_0^\infty e^{-\lambda x} - e^{-\lambda x(1+y_2)} \mathrm{d}x \\
&= 1 - \frac{1}{1+y_2} \\
\implies f_{Y_2}(y_2) &= \frac{1}{(1+y_2)^2}
\end{aligned}
$$

As the joint distribution of X_1 and X_2 is $F_{(X_1,X_2)}(x_1,x_2) = \lambda^2 e^{-\lambda(x_1+x_2)}$ and $Y_1 = X_1 + X_2; Y_2 = \frac{X_1}{x_2}$; thus the transformation Jacobian is

$$
J = \begin{bmatrix} 1 & 1 \\ \frac{1}{X_2} & -\frac{X_1}{X_2^2} \end{bmatrix}
$$

Thus, the absolute determinent of the Jacobian is $|J| = \frac{X_1+X_2}{X_2^2} = \frac{Y_1}{X_2^2}$.
As $Y_2 = \frac{X_1}{X_2} = \frac{X_1+X_2-X_2}{X_2} = \frac{Y_1}{X_2} - 1 \implies X_2 = \frac{Y_1}{Y_2+1}$.
Thus, $|J| = \frac{(Y_2+1)^2}{Y_1}$. As $X_1 = \frac{Y_1 Y_2}{Y_2+1}$; the joint distribution of Y_1 and Y_2 is

$$
\begin{aligned}
f_{(Y_1,Y_2)}(y_1,y_2) &= \frac{1}{|J|} f_{(X_1,X_2)}(x_1,x_2) \\
&= \frac{y_1}{(y_2+1)^2} \lambda^2 e^{-\lambda y_1} \\
&= \lambda^2 y_1 e^{-\lambda y_1} \frac{1}{(y_2+1)^2} \\
&= f_{Y_1}(y_1) f_{Y_2}(y_2)
\end{aligned}
$$

Thus, Y_1 and Y_2 are independent. □