Analysis I, HW 1 Solution Notes

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This is a compilation of solutions for Assignment 1 of Probability Theory class instructed by Prof. R. Srinivasan. The solutions are of original design. The template is a modified template of the one used for USAMO solutions by Evan Chen.

Contents

§0 Problems

1. If A, B, C are arbitrary sets, show, for any probability measure P, that

(i) $\mathbb{P}(A \cap B \cap C) \leq \mathbb{P}(A) \wedge \mathbb{P}(B) \wedge \mathbb{P}(C)$.

(ii) $\mathbb{P}(A \cup B \cup C) \geq \mathbb{P}(A) \vee \mathbb{P}(B) \vee \mathbb{P}(C)$.

(iii) $\mathbb{P}(A \cap B) \geq \mathbb{P}(A) + \mathbb{P}(B) - 1$.

2. An urn contains n balls numbered $1, 2, \ldots, n$. We select at random r balls, (a) with replacement, (b) without replacement. What is the probability that the largest selected number is m?

3. A and B throw alternately a pair of dice in that order. A wins if he scores 6 points before B gets 7 points, in which case B wins. If A starts the game what is his probability of winning?

4. Urn A contains w_1 white balls and b_1 black balls. Urn B contains w_1 white balls and w_2 black balls. A ball is drawn from A and is placed into B, and then a ball is transferred from B to A. Finally, a ball is selected from A. What is the probability that the ball will be white?

5. Let L denotes the space of all integer valued random variables on a probability space. We define $X \sim Y$ if $P(\{X = Y\}) = 1$ and $\tilde{L}$ denotes the vector space $L/\sim$. Define

$$d([X], [Y]) = \sum_{n \in \mathbb{Z}} |P(\{X = n\}) - P(\{Y = n\})|,$$

where $X \in [X]$, $Y \in [Y]$ for some element in the equivalence class. Show that d is a well-defined metric on $\tilde{L}$ and that

$$d([X], [Y]) = 2 \sup_{A \subseteq \mathbb{Z}} |P(\{X \in A\}) - P(\{Y \in A\})|.$$

6. Find the most probable value of a random variable following Poisn(λ)

7. A random number N of dice is thrown. Let A_i be the event that $\{N = i\}$, and assume that $\mathbb{P}(A_i) = 2^{-i}$, $i \geq 1$. The sum of the scores is the random variable S. Find the probability that:

(a) $\{N = 2\}$ given $\{S = 4\}$

(b) $\{S = 4\}$ given N is even

(c) $\{N = 2\}$, given that $\{S = 4\}$ and the first die showed 1

(d) the largest number shown by any die is r, where S is unknown

8. Let X and Y be independent random variables taking values in the positive integers and having the same geometric mass function with parameter p. Find:

(i) $\mathbb{P}(Y > X)$

(ii) $\mathbb{P}(X = Y)$

(iii) $\mathbb{P}(X \geq kY)$ for $k \in \mathbb{Z}^+$

(iv) $\mathbb{P}(Y \pmod{X} = 0)$

(v) $\mathbb{P}(x = rY)$ for $r \in \mathbb{Q}^+$

9. Suppose a building has f number of floors above the basement, where it starts with N passengers. Assume that the passengers get off, independently of each other, at one of the f floors with an equal probability. Let X be the first floor after the basement at which the elevator stops to let a passenger off. Compute $E(X)$.

10. Find the marginal mass functions of the multinomial distribution

11. Let X and Y be independent geometric random variables with respective parameters p_1 and p_2. Show that the mass function of $X+Y$ is given by

$$f_{X+Y}(k) = \frac{p_1 p_2}{p_1 - p_2}((1-p_2)^{k-1} - (1-p_1)^{k-1})$$

12. Let N be Poisson distributed with parameter λ Show that, for any function g (such that the following expectations exist),

$$\mathbb{E}(Ng(N)) = \lambda \mathbb{E}(g(N+1))$$

13. Find the generating function of a random variable with mass function $f(m) = \frac{1}{m(m+1)}$ with $m \geq 1$.

14. Find the generating function of $f(m) = \frac{(1-p)p^{|m|}}{(1+p)}, m \in \mathbb{Z}$, where $0 < p < 1$. Find the mean and variance.

§1 Solutions

§1.1 Problem 1

Problem statement

If A, B, C are arbitrary sets, show, for any probability measure P, that

(i) $\mathbb{P}(A \cap B \cap C) \leq \mathbb{P}(A) \wedge \mathbb{P}(B) \wedge \mathbb{P}(C)$.

(ii) $\mathbb{P}(A \cup B \cup C) \geq \mathbb{P}(A) \vee \mathbb{P}(B) \vee \mathbb{P}(C)$.

(iii) $\mathbb{P}(A \cap B) \geq \mathbb{P}(A) + \mathbb{P}(B) - 1$.

Lemma 1.1

$$A \supseteq B \implies \mathbb{P}(A) \geq \mathbb{P}(B)$$

Proof. Proof. $A = B + B/A$ where $B \cap B/A = \phi$.
Thus, $\mathbb{P}(A) = \mathbb{P}(B + B/A) = \mathbb{P}(B) + \mathbb{P}(B/A) \geq \mathbb{P}(B)$ □

This automatically solves (i) and (ii) as $A \cap B \cap C \subseteq A, B, C$ and $A \cup B \cup C \supseteq A, B, C$. For (iii), we use $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$ to get

$$\begin{aligned}\mathbb{P}(A \cap B) &\geq \mathbb{P}(A) + \mathbb{P}(B) - 1 \\ 1 &\geq \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B) \\ 1 &\geq \mathbb{P}(A \cup B)\end{aligned}$$

Which is true by the definition of probability. □

§1.2 Problem 2

Problem statement

An urn contains n balls numbered $1, 2, \ldots, n$. We select at random r balls, (a) with replacement, (b) without replacement. What is the probability that the largest selected number is m?

Proof. Let the largest selected number be represented by the random variable X. (a)

$$\mathbb{P}(X = m) = \frac{m^r - (m-1)^r}{n^r}$$

As we want atleast one ball to be equal to m and don't really care about others. So m^r over counts the cases where none of the balls are greater than $m - 1$. So by subtracting, $(m-1)^r$, we are done. (b)

$$\mathbb{P}(X = m) = \begin{cases} \frac{\binom{m-1}{r-1}}{\binom{n}{r}} & m \geq r \\ 0 & \text{otherwise} \end{cases}$$

This is as we want all the selected balls from $\{1, 2, \ldots, m-1\}$ and will fail to do so if number of draws is larger than $m-1$. □

§1.3 Problem 3

Problem statement

A and B throw alternately a pair of dice in that order. A wins if he scores 6 points before B gets 7 points, in which case B wins. If A starts the game what is his probability of winning?

Proof. Notice,

$$\mathbb{P}(2d6 = 6) = \frac{5}{36}$$
$$\mathbb{P}(2d6 = 7) = \frac{1}{6}$$
$$\therefore \mathbb{P}(\text{A win}) = \frac{5}{36} + \frac{5}{36}\frac{5}{6}\frac{31}{36} + \ldots$$
$$= \frac{5}{36}\left(1 + \frac{5}{6}\frac{31}{36} + \frac{25}{36}\frac{31^2}{36^2} + \ldots\right)$$
$$= \frac{5}{36}\frac{1}{1 - \frac{5}{6}\frac{31}{36}} = \frac{5}{36}\frac{1}{1 - \frac{155}{216}}$$
$$= \frac{5}{36}\frac{216}{61} = \frac{30}{61}$$

□

Remark 1.2. Before this question was clarified, here is my orignal solution: The game can atmost run for 5 turns as $2 * 3 = 6$ and thus, the game will surely end on the 5^{th} turn.
Secondly, $\mathbb{P}(A\text{wins}) = 1 - \mathbb{P}(B\text{wins})$.
We shall compute the latter as it is easier to do so.
Round 1
A can fail to win this round if A rolls $2, 3, 4, 5$ which has a probability $\frac{10}{36}$.
Round 2
This can occur if B rolls $2, 3, 4, 5, 6$ which has a probability of $\frac{15}{36}$.
This also implies that the probability B wins on round 2 is $\frac{21}{36}$
Round 3 If A rolled a $4, 5$, this round ends the game.
The only case the game continues is if A rolled a 2 in round 1 and now rolls $2, 3$ or if A rolled a 3 in round 1 and now rolls 2.
This has probability $\frac{1}{10}\frac{3}{36} + \frac{2}{10}\frac{1}{36}$.
Round 4 B wins here with probability $\frac{1}{15}\frac{30}{36} + \frac{2}{15}\frac{33}{36} + \frac{3}{15}\frac{35}{36} + \frac{9}{15}$.

Thus, the probability of B winning is

$$\frac{10}{36}\frac{21}{36} + \frac{10}{36}\frac{15}{36}\left(\frac{1}{10}\frac{3}{36} + \frac{2}{10}\frac{1}{36}\right)\left(\frac{1}{15}\frac{30}{36} + \frac{2}{15}\frac{33}{36} + \frac{3}{15}\frac{35}{36} + \frac{9}{15}\right) = \frac{91595}{559872}$$

Thus,

$$\mathbb{P}(A \text{ win}) = \frac{468277}{559872}$$

§1.4 Problem 4

Problem statement

Urn A contains w_1 white balls and b_1 black balls. Urn B contains w_1 white balls and w_2 black balls. A ball is drawn from A and is placed into B, and then a ball is transferred from B to A. Finally, a ball is selected from A. What is the probability that the ball will be white?

Proof. We have three cases. A has $w_1 - 1, w_1, w_1 + 1$ white balls. The first occurs if we drew a white then a black, the second if we drew a white and white or a black and black and third if we drew a black then a white.
Thus,

$$\begin{aligned}\mathbb{P}(d_3 = W) = \frac{w_1 - 1}{w_1 + b_1}\left(\frac{w_1}{w_1 + b_1}\frac{w_2}{w_1 + w_2 + 1}\right) \\ + \frac{w_1}{w_1 + b_1}\left(\frac{w_1}{w_1 + b_1}\frac{w_1 + 1}{w_1 + w_2 + 1} + \frac{b_1}{w_1 + b_1}\frac{w_2 + 1}{w_1 + w_2 + 1}\right) \\ + \frac{w_1 + 1}{w_1 + b_1}\left(\frac{b_1}{w_1 + b_1}\frac{w_1}{w_1 + w_2 + 1}\right)\end{aligned}$$

□

§1.5 Problem 5

Problem statement

Let L denotes the space of all integer valued random variables on a probability space. We define $X \sim Y$ if $P(\{X = Y\}) = 1$ and $\tilde{L}$ denotes the vector space $L/\sim$. Define

$$d([X], [Y]) = \sum_{n \in \mathbb{Z}} |P(\{X = n\}) - P(\{Y = n\})|,$$

where $X \in [X]$, $Y \in [Y]$ for some element in the equivalence class. Show that d is a well-defined metric on $\tilde{L}$ and that

$$d([X], [Y]) = 2 \sup_{A \subseteq \mathbb{Z}} |P(\{X \in A\}) - P(\{Y \in A\})|.$$

Proof. Notice

$$d([X], [X]) = \sum_{n \in \mathbb{Z}} |\mathbb{P}(\{X = n\}) - \mathbb{P}(\{X = n\})| = 0$$

As the summation only has non-negative parts, $d([X], [Y]) = 0 \implies \mathbb{P}(\{X = n\}) = \mathbb{P}(\{Y = n\} \forall n \in \mathbb{Z}$ which means X is sthocastically identical to Y.

$$d([X], [Y]) = \sum_{n \in \mathbb{Z}} |\mathbb{P}(\{X = n\}) - \mathbb{P}(\{Y = n\})| = \sum_{n \in \mathbb{Z}} |\mathbb{P}(\{Y = n\}) - \mathbb{P}(\{X = n\})| = d([Y], [X])$$

$$d([X],[Y])+d([Y],[Z]) = \sum_{n\in\mathbb{Z}} |\mathbb{P}(\{X=n\})-\mathbb{P}(\{Y=n\})| + \sum_{n\in\mathbb{Z}} |\mathbb{P}(\{Y=n\})-\mathbb{P}(\{Z=n\})|$$
$$> \sum_{n\in\mathbb{Z}} |\mathbb{P}(\{X=n\}) - \mathbb{P}(\{Z=n\})| = d([x],[Z])$$

Thus, the above defined is a valid metric.
We define $A = \{k|\mathbb{P}(X=k) \geq \mathbb{P}(Y=k\}$. This is trivially the supremum set for the above property as including any other element contradicts the maximality of supremum.
Also notice, $\mathbb{P}(X \in A) + \mathbb{P}(X \in A^c) = 1$ for all A as X is a random variable.
Thus, for our A,

$$d([X],[Y]) = \sum_{A\subseteq\mathbb{Z}} \mathbb{P}(\{X\in A\}) - \mathbb{P}(\{Y\in A\}) + \sum_{A\subseteq\mathbb{Z}} \mathbb{P}(\{Y\in A\}) - \mathbb{P}(\{X\in A\})$$

Using $\mathbb{P}(X \in A) + \mathbb{P}(X \in A^c) = \mathbb{P}(Y \in A) + \mathbb{P}(Y \in A^c) \implies \mathbb{P}(X \in A) - \mathbb{P}(Y \in A) = \mathbb{P}(Y \in A^c) - \mathbb{P}(X \in A^c)$

$$= 2\sum_{A\subseteq\mathbb{Z}} |\mathbb{P}(\{X\in A\}) - \mathbb{P}(\{Y\in A\})|$$

Using the fact that A is the supremum

$$d([X],[Y]) = 2\sup_{A\subseteq\mathbb{Z}} |\mathbb{P}(\{X\in A\}) - \mathbb{P}(\{Y\in A\})|$$

And we are done! □

§1.6 Problem 6

Problem statement

Find the most probable value of a random variable following Poisn(λ)

Proof. Let X be poisson distributed with parameter λ, Then,

$$\frac{\mathbb{P}(x=k+1)}{\mathbb{P}(x=k)} = \frac{\frac{e^{-\lambda}\lambda^{k+1}}{(k+1)!}}{\frac{e^{-\lambda}\lambda^{k}}{k!}} = \frac{\lambda}{k+1}$$

This implies $\mathbb{P}(x=k+1) > \mathbb{P}(X=k)$ as long as $\lambda > k+1$.
For $0 < \lambda < 1$, the mode is realized at $\mathbb{P}(x=0)$ as $\lambda < k+1$ for all $k > 0$ and the probability is decreasing.
For $\lambda > 1$ and is not an integer, the mode is realized at $\mathbb{P}(x = \lfloor x \rfloor)$ as $\lambda < k+1$ for all $k \geq \lfloor \lambda \rfloor$.
For $\lambda > 1$ and is an integer, the mode is realized at $\mathbb{P}(x=\lambda)$ and $\mathbb{P}(x=\lambda-1)$ as $\lambda < \lambda+1$ for all $k > \lambda$. Here however, at $k = \lambda - 1$ we get $\mathbb{P}(x=k+1) = \mathbb{P}(x=k)$ and thus, we have two modes. □

§1.7 Problem 7

Problem statement

A random number N of dice is thrown. Let A_i be the event that $\{N = i\}$, and assume that $\mathbb{P}(A_i) = 2^{-i}$, $i \geq 1$. The sum of the scores is the random variable S. Find the probability that:

(a) $\{N = 2\}$ given $\{S = 4\}$

(b) $\{S = 4\}$ given N is even

(c) $\{N = 2\}$, given that $\{S = 4\}$ and the first die showed 1

(d) the largest number shown by any die is r, where S is unknown

Proof. (a) Using Bayes and the fact that for $N \geq 5$ $S \geq 5$.

$$\begin{aligned}\mathbb{P}(N = 2|S = 4) &= \frac{\mathbb{P}(S = 4|N = 2)\mathbb{P}(N = 2)}{\mathbb{P}(S = 4)} \\ &= \frac{\frac{1}{12}\frac{1}{4}}{\frac{1}{6}\frac{1}{2} + \frac{1}{12}\frac{1}{4} + \frac{1}{72}\frac{1}{8} + \frac{1}{1296}\frac{1}{16}} \\ &= \frac{432}{2197}\end{aligned}$$

(b)

$$\mathbb{P}(S = 4|N \pmod 2 \equiv 0) = \frac{\mathbb{P}(S = 4 \cap S \in \{2, 4\})}{2^{-2} + 2^{-4} + \dots}$$

Using $\mathbb{P}(A \cap B) = \mathbb{P}(A|B)\mathbb{P}(B)$

$$= \frac{\frac{1}{12}\frac{1}{4} + \frac{1}{1296}\frac{1}{16}}{\frac{1}{3}} = \frac{433}{6912}$$

(c) Let the first dice be the random variable d_1. Thus, we wish to solve for $\mathbb{P}(N = 2|S = 4, d_1 = 1)$.
As the dice rolls are independent of each other, we will define $N' = N - 1$.
Now we have $\mathbb{P}(N = 2|S = 4, d_1 = 1) = \mathbb{P}(N' = 1|S = 3)$
Using Bayes,

$$\begin{aligned}\mathbb{P}(N' = 1|S = 3) &= \frac{\mathbb{P}(S = 3|N' = 1)\mathbb{P}(N' = 1)}{\mathbb{P}(S = 3)} \\ &= \frac{\frac{1}{6}\frac{1}{4}}{0\frac{1}{2} + \frac{1}{6}\frac{1}{4} + \frac{1}{18}\frac{1}{8} + \frac{1}{216}\frac{1}{16}} \qquad = \frac{144}{169}\end{aligned}$$

(d) We are trying to solve for $\mathbb{P}(\max(d) = r)$ for a given r. For $r < 1$ or $r > 6$; we trivally report 0. Now we consider $r \in \{1, 2, 3, 4, 5, 6\}$
We will first try to solve, For a given N $\mathbb{P}(\max(d) = r|N = n)$ for any n.
Here all the n dice can take r possible values to not violate this. However, what if all dice are below r? Thus, we must subtract the case where all n dice take values from $1 - r - 1$.
Thus,

$$\mathbb{P}(\max(d) = r|N = n) = \frac{r^n - (r-1)^n}{6^n}$$

Using law of alternatives,

$$\begin{aligned}
\mathbb{P}(\max(d) = r) &= \sum_{n=1}^{\infty} \mathbb{P}(\max(d) = r | N = n) \\
&= \sum_{n=1}^{\infty} \frac{r^n - (r-1)^n}{6^n} \frac{1}{2^n} \\
&= \sum_{n=1}^{\infty} \left(\frac{r}{12}\right)^n - \sum_{n=1}^{\infty} \left(\frac{r-1}{12}\right)^n \\
&= \frac{\frac{r}{12}}{1 - \frac{r}{12}} - \frac{\frac{r-1}{12}}{1 - \frac{r-1}{12}} \\
&= \frac{r}{12 - r} - \frac{r-1}{13 - r} \\
&= \frac{12}{(12-r)(13-r)}
\end{aligned}$$

□

§1.8 Problem 8

Problem statement

Let X and Y be independent random variables taking values in the positive integers and having the same geometric mass function with parameter p. Find:

(i) $\mathbb{P}(Y > X)$

(ii) $\mathbb{P}(X = Y)$

(iii) $\mathbb{P}(X \geq kY)$ for $k \in \mathbb{Z}^+$

(iv) $\mathbb{P}(Y \pmod X) = 0)$

(v) $\mathbb{P}(x = rY)$ for $r \in \mathbb{Q}^+$

Proof. (i) Consider $\mathbb{P}(Y > X | X = n)$.
This is clearly equal to $\mathbb{P}(Y > n) = q^n p + q^{n+1} p + \cdots = q^n p(1 + q + q^2 + \ldots) = q^n p \frac{1}{1-q} = q^n = (1-p)^n$ where $q = 1 - p$. Using law of alternatives,

$$\begin{aligned}
\mathbb{P}(Y > X) &= \sum_{i=1}^{\infty} \mathbb{P}(Y > X | X = i) \mathbb{P}(X = i) \\
&= \sum_{i=1}^{\infty} (1-p)^i (1-p)^{i-1} p \qquad = p \sum_{i=1}^{\infty} (1-p)^{2i-1} = p \frac{1-p}{1-(1-p)^2} \\
&= \frac{p - p^2}{1 - 1 + 2p - p^2} \qquad = \frac{p-p^2}{2p - p^2} \\
&= \frac{1-p}{2-p}
\end{aligned}$$

(ii) $1 = \mathbb{P}(X > Y) + \mathbb{P}(X = Y) + \mathbb{P}(X < Y) = \frac{1-p}{2-p} + \mathbb{P}(X = Y) + \frac{1-p}{2-p}$
$\implies \mathbb{P}(X = Y) = 1 - \frac{2-2p}{2-p} = \frac{2-p-2+2p}{2-p} = \frac{p}{2-p}$

(iii) For $Y = n$,

$$\mathbb{P}(X \geq kY|Y = n)$$
$$= \mathbb{P}(X \geq kn) \quad = q^{kn-1}p + q^{kn}p + \dots = q^{kn-1}p(1 + q + q^2 + \dots) \quad = q^{kn-1} = (1-p)^{kn-}$$

Thus, using law of alternatives

$$\mathbb{P}(X > Y) = \sum_{i=1}^{\infty} P(X \geq kY|Y = i)P(Y = i) = (1-p)^{ki-1}q^{i-1}p = p\frac{(1-p)^{k-1}}{1-(1-p)^{k+1}}$$

(iv) For $X = n$,

$$\mathbb{P}(Y \pmod X) = 0|X = n)$$
$$= \mathbb{P}(Y = n) + \mathbb{P}(Y = 2n) + \dots$$
$$= q^{n-1}p + q^{2n-1}p + \dots$$
$$= \frac{p}{q}(q^n + q^{2n} + \dots)$$
$$= \frac{p}{q}\frac{q^n}{1-q^n}$$

Using law of alternatives,

$$\mathbb{P}(Y \pmod X) = 0) = \sum_{i=1}^{\infty} \mathbb{P}(Y \pmod X) = 0|X = i)\mathbb{P}(X = i)$$
$$= \sum_{i=1}^{\infty} \frac{p}{q}\frac{q^i}{1-q^i}q^{i-1}p$$
$$\frac{p^2}{q^2}\sum_{i=1}^{\infty}\frac{q^{2i}}{1-q^i}$$

(v) Let $r = \frac{a}{b}$ in its lowest form.
Consider $\mathbb{P}(X = rY|Y = k)$.
This is clearly 0 if $k \pmod b \not\equiv 0$. Let $k = nb$.

$$\mathbb{P}(X = rY|Y = k) = \mathbb{P}(X = na|Y = nb) = pq^{na-1}$$

From the above computation.
Thus, using law of alternative,

$$\mathbb{P}(X = rY) = \sum_{i=0}^{\infty} \mathbb{P}(X = rY|Y = ib)\mathbb{P}(Y = ib)$$
$$= \sum_{i=0}^{\infty} pq^{ia-1}pq^{ib-1}$$
$$= \frac{p^2}{q^2}\sum_{i=0}^{\infty} q^{(a+b)i}$$
$$= \frac{p^2}{q^2}\frac{1}{1-q^{a+b}}$$

□

§1.9 Problem 9

Problem statement

Suppose a building has f number of floors above the basement, where it starts with N passengers. Assume that the passengers get off, independently of each other, at one of the f floors with an equal probability. Let X be the first floor after the basement at which the elevator stops to let a passenger off. Compute $E(X)$.

Proof. Let's compute $\mathbb{P}(X = k)$.
If $k > f$, it clearly makes no sense and hence, the probability will be 0.
If $k \leq f$, There are $f - k + 1$ ways for every passenger to stop at or beyond k; however, we need to subtract where everyone gets off beyond k. Using the uniformity,

$$\mathbb{P}(x = k) = \frac{(f - k + 1)^N - (f - k)^N}{f^N}$$

Thus,

$$\begin{aligned}
E(X) &= \sum_{i=1}^{\infty} i\mathbb{P}(X = i) \\
&= \sum_{i=1}^{f} i\frac{(f - i + 1)^N - (f - i)^N}{f^N} \\
&= \frac{1}{f^N}\sum_{i=1}^{f} i(f - i + 1)^N - \sum_{i=1}^{f} i(f - i)^N \\
&= \frac{1}{f^N}(1f^N + 2(f - 1)^N + \cdots + (f - 1)2^N + f1^N - 1(f - 1)^N - 2(f - 2)^N - \cdots - (f - 1)1^N - f0^N) \\
&= \frac{1}{f^N}(f^N + (f - 1)^N + (f - 2)^N + \cdots + 1^N) \\
&= \frac{1}{f^N}\sum_{i=1}^{f} i^N
\end{aligned}$$

□

§1.10 Problem 10

Problem statement

Find the marginal mass functions of the multinomial distribution

Proof. The multinational distribution is defined by the sum

$$\text{Multi}(n, p_1, p_2, \ldots, p_n) = \sum_{i=1}^{n} \text{Bernouli}(n, p_1, p_2, \ldots, p_n)$$

Where Bernouli(n) is a probability distribution over binary vectors of length n with a single 1.

The probability that the i^{th} entry is 1 is p_i and $\sum_{i=1}^{n} p_i = 1$. To find only $\mathbb{P}(X_i = k)$, we can look at the Bernouli sums.
We want exactly k of the n Bernouli random vectors to have 1 in the i^{th} spot and others to have it elsewhere.
This has probability $\frac{n}{k} p_i^k (1-p_i)^{n-k}$ which is $\text{binom}(n, p)$.
Thus, the marginal of multinational is the binomial distribution.

□

§1.11 Problem 11

Problem statement

Let X and Y be independent geometric random variables with respective parameters p_1 and p_2. Show that the mass function of $X+Y$ is given by

$$f_{X+Y}(k) = \frac{p_1 p_2}{p_1 - p_2}((1-p_2)^{k-1} - (1-p_1)^{k-1})$$

Proof. Using Law of Alternatives,

$$f_{X+Y}(k) = \sum_{i=1}^{k-1} \mathbb{P}(X = i)\mathbb{P}(Y = k-i)$$

$$= \sum_{i=1}^{k-1} q_1^{i-1} p_1 q_2^{k-i-1} p_2$$

$$= \frac{p_1}{q_1}\frac{p_2}{q_2} q_2^k \sum_{i=1}^{k-1} \left(\frac{q_1}{q_2}\right)^i$$

$$= \frac{p_1}{q_1}\frac{p_2}{q_2} q_2^k \frac{\left(\frac{q_1}{q_2}\right)^k - \frac{q_1}{q_2}}{\frac{q_1}{q_2} - 1}$$

$$= \frac{p_1 p_2}{q_1 q_2} q_2^{k+1} \frac{q_1^k - q_1 q_2^{k-1}}{q_1 q_2^k - q_2^{k+1}}$$

$$= p_1 p_2 \frac{q_1^{k-1} - q_2^{k-1}}{q_1 - q_2}$$

$$= \frac{p_1 p_2}{1 - p_1 - 1 + p_2}((1-p_1)^{k-1} - (1-p_2)^{k-1})$$

$$= \frac{p_1 p_2}{p_1 - p_2}((1-p_2)^{k-1} - (1-p_1)^{k-1})$$

□

§1.12 Problem 12

Problem statement

Let N be Poisson distributed with parameter λ Show that, for any function g (such that the following expectations exist),

$$\mathbb{E}(Ng(N)) = \lambda\mathbb{E}(g(N+1))$$

Proof. Using the definition of Expectation.

$$\begin{aligned}
&\mathbb{E}(Ng(N)) \\
&\qquad = \sum_{i=1}^{\infty} \frac{e^{-\lambda}\lambda^i}{i!} ig(i) \\
&\qquad = \lambda \sum_{i=1}^{\infty} \frac{e^{-\lambda}\lambda^{i-1}}{(i-1)!} g(i) \\
&\qquad = \lambda \sum_{i=0}^{\infty} \frac{e^{-\lambda}\lambda^{i}}{i!} g(i+1) \\
&\qquad = \lambda\mathbb{E}(g(i+1))
\end{aligned}$$

□

§1.13 Problem 13

Problem statement

Find the generating function of a random variable with mass function $f(m) = \frac{1}{m(m+1)}$ with $m \geq 1$.

Proof. Using the fact that probability of $m = i$ is only defined for $i \geq 1$,

$$\begin{aligned}
\Phi_m(t) &= \sum_{i=1}^{\infty} \mathbb{P}(m = i)t^i \\
&= \sum_{i=1}^{\infty} \frac{t^i}{i(i+1)} \\
&= \sum_{i=1}^{\infty} \frac{((i+1)-i)t^i}{i(i+1)} \\
&= \sum_{i=1}^{\infty} \frac{t^i}{i} - \sum_{i=1}^{\infty} \frac{t^i}{i+1} \\
&= -\ln(1-t) - \frac{1}{t}\sum_{i=1}^{\infty} \frac{t^{i+1}}{i+1} \\
&= -\ln(1-t) - \frac{1}{t}\left(-t + t + \sum_{i=1}^{\infty} \frac{t^{i+1}}{i+1}\right)
\end{aligned}$$

$$= -\ln(1-t) - \frac{1}{t}\left(-t - \ln(1-t)\right)$$
$$= \ln(1-t)\left(\frac{1}{t} - 1\right) + 1$$

We could do all the operations and substitutions as $|t| \leq 1$ and $\sum_{i=1}^{\infty} \frac{((i+1)-i)t^i}{i(i+1)}$ converges absolutely by ratio test as $\frac{tn}{(n+2)} \to 0$ as $n \to 0$.

□

§1.14 Problem 14

Problem statement

Find the generating function of $f(m) = \frac{(1-p)p^{|m|}}{(1+p)}, m \in \mathbb{Z}$, where $0 < p < 1$. Find the mean and variance.

Proof.

$$\Phi_m(t) = \sum_{i \in \mathbb{Z}} t^i f(i)$$
$$= \frac{1-p}{p} \sum_{i \in \mathbb{Z}} t^i p^{|i|}$$

We will now use $\mathbb{E}(m) = \Phi'_m(1)$ to get

$$\mathbb{E}(m) = \left(\frac{1-p}{p} \sum_{i \in \mathbb{Z}} t^i p^{|i|}\right)'(1) = \frac{1-p}{p} \sum_{i \in \mathbb{Z}} 1^{i-1} i p^{|i|}$$

As for every $i \neq 0$, $ip^{|i|} + (-i)p^{|-i|} = 0$. Thus,

$$\mathbb{E}(m) = \Phi'_m(1) = 0$$

Now using $\sigma^2(m) = \Phi''_m(1) + \Phi'_m(1) - (\Phi'_m(1))^2$.
As $\Phi'_m(1) = 0, \sigma^2(m) = \Phi''_m(1)$. As $p \leq 1 \leq \frac{1}{p}$, the power series converges.

$$\Phi_m(t) = \frac{1-p}{p}\left(\sum_{i=0}^{\infty}(pt)^i + \sum_{i=1}^{\infty}\left(\frac{p}{t}\right)^i\right) = \frac{1-p}{p}\left(\frac{1}{1-pt} + \frac{\frac{p}{t}}{1-\frac{p}{t}}\right) = \frac{(1-p)^2(1+p)t}{p(t-p)(1-tp)}$$

Thus, $\Phi''_m(t) = \frac{2(1-p)^2(1+p)(1+p^2+pt(t^2-3))}{(t-p)^3(1-pt)^3}$.
Thus, $\Phi''_m(t) = \frac{2(1-p)^2(1+p)(1+p^2-2p)}{(1-p)^3(1-p)^3} = \frac{2(1+p)}{(1-p)^2}$.
Thus, $\sigma^2(m) = \frac{2(1+p)}{(1-p)^2}$ □