CHENNAI MATHEMATICAL INSTITUTE

Probability Theory, B.Sc. I, 2025, Jan-April

End-term Examination.

You can score a maximum of 100 marks, combined from both sections A and B. 100

Part A

The problems in this section are either already discussed in the class or a slight variation of that. Give the required proof completely. You can score up to a maximum of 45 marks from this section.

1. Let X be a non-negative continuous random variable with distribution function F. Show that
$$E(X^r) = \int_0^\infty r x^{r-1}(1 - F(x))dx,$$
for which the above expectation is finite. (You get half of the marks for proving the case $r = 1$.) 15

2. Let $X \sim Unif[a, b]$ and F be a distribution function supported in $[a, b]$. Construct a random variable (with proof!) Y with distribution F. 10

3. Compute the characteristic function of the standard normal distribution $N(0, 1)$. 15

4. Let X be a random variable with density function f and characteristic function ϕ. Further assume that f is continuous and $\int_{-\infty}^{\infty} |\phi(t)|dt < \infty$. Show for all $x \in \mathbb{R}$ that
$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx}\phi(t)dt.$$
15

5. Let $\{X_n\}_{n=1}^{\infty}, X$ be continuous random variables with characteristic functions $\{\phi_n\}_{n=1}^{\infty}, \phi$ respectively. Show, if $\phi_n(t) \mapsto \phi(t)$ for all $t \in \mathbb{R}$ then that $X_n \mapsto X$ in distribution. (Partial marks if you assume some of the lemmas without proof.) 30

Part B

You may use any result proved in the class. You can score up to a maximum of 70 marks from this section.

1. Let X and Y have the bivariate normal distribution with means 0 and variances 1 and covariance ρ. Find the joint density function of $X + Y$ and $X - Y$, and their marginal density functions. 20

2. Let X and Y be independent (continuous) random variables with common distribution function F and density function f. Compute the densities of $\max\{X, Y\}$ and $\min\{X, Y\}$, assuming that f is continuous. $2F(x)f(x)$ $2f(x)(1-F(x))$ 10

3. Prove the answer in the last problem without assuming continuity. 15

4. Let $\{X_n : n \in \mathbb{N}\}$ be independent and identically distributed family of random variables with with distribution function F satisfying $F(y) < 1$ for all $y \in \mathbb{R}$. For $x \in \mathbb{R}$, let $Y_x = \min\{k : X_k > x\}$. Compute the distribution of Y_x and its expectation. $1-(F(x))^k$ $\frac{1}{1-F(x)}$ 25

5. Let X and Y have the joint density $f(x, y) = cx(y - x)e^{-y}$, for $0 \le x \le y < \infty$. Find (i) c (ii) $E(X/Y)$ (iii) $E(Y/X)$. $xe^{-x}(x+2)$ 15

6. Let the continuous random variables X and Y have joint density function

$$f(x, y) = \frac{1}{x}, \quad 0 \le y \le x \le 1,$$
$$= 0, \quad \text{otherwise},$$

Find the probability of $\{X^2 + Y^2 \le 1\}$. $2\ln(\sqrt{2}+1)$ $\sqrt{x^2+y^2}$, $\tan^{-1}(\frac{y}{x})$ 20

7. Let X be a random variable with density function f and characteristic function ϕ. Show (subject to an appropriate condition on f) that $\int_{-\infty}^{\infty} |\phi(t)|^2 dt = \int_{-\infty}^{\infty} f(x)^2 dx$. (Hint: Notice that $|\phi|^2$ is a characteristic function of some random variable.) 20

8. Let X has the $\Gamma(1, s)$ distribution. Let Y be such that the conditional distribution of Y given $X = x$ follows Poisson with parameter x. Find the characteristic function of Y and show that $\frac{1}{(2-e^{it})^s}$ $Var(E(Y|X)) + E(Var(Y|X))$

$$\frac{Y - E(Y)}{\sqrt{Var(Y)}} \mapsto N(0, 1), \quad \text{in distribution as } s \mapsto \infty.$$

(Use continuity theorem along the lines of the proof of CLT.) 30

$\frac{\alpha^k t^{k-1} e^{-\alpha t}}{\Gamma(k)} = f(x)$ $\Gamma(k) = \int_0^\infty t^{k-1} e^{-t} dt$

$P(X|Y) = \frac{P(X \cap Y)}{P(Y)}$

$\left(\frac{1}{2}\right)^{y+s} \binom{y+s-1}{y}$ $0 \le y \le x \le 1$

$\frac{\ln(n-v)}{4\sqrt{v}}$ x^2+y^2