CHENNAI MATHEMATICAL INSTITUTE

Probability Theory, B.Sc. I, 2025, Jan-April

Mid-term Examination.

You can score a maximum of 100 marks, combined from both sections A and B.

Part A

The problems in this section are either already discussed in the class or a slight variation of that. Give the required proof completely. You can score up to a maximum of 45 marks from this section.

1. Show that a non-negative integer valued random variable X satisfies

$$P(\{X = n + k / X \geqslant n\}) = P(\{X = k\}), \quad \forall k, n \geq 0$$

if and only if it is geometric. 15

2. Show that a random variable X with hypergeometric distribution with parameter (N, n, k), can be written as sum of k dependent Bernoulli random variables. Using that calculate the mean and variance of X. 15

3. Let X be a random variable having a negative binomial distribution with parameters α and p. Derive the generating function of X. Find the mean and variance of X. 15

4. State and prove the strong law of large numbers. 15

5. A hen lays N eggs, where N has the Poison distribution with parameter λ. Each egg hatches with probability p independently of the other eggs. Let K be the number of chicks. Find the mass function of K. Also find $E(K/N)$, $E(K)$ and $E(N/K)$. 15

$\sum_{n=0}^{\infty} \sum_{y=0}^{\infty} y^2 P(Y|X)\ P(X=n)$ $E(Y \cdot E(Y|X)) - E(Y)E(Y)$

$E(y^2)$ ccc... a

$P(A) = P(A)\ \sum P(A \cap B) + \sum P(A \cap B \cap C)$

$P(A) - P(A \cap B)$

Part B

You may use any result proved in the class. You can score up to a maximum of 70 marks from this section.

$p - (n-1)q + \binom{n-1}{2}x$

a, ca, cca,

1. Let $\Omega = \{a, b, c\}$ represents the sample space of a random experiment. Suppose we repeat this experiment indefinitely and independently. Calculate, in terms of $p = P(\{a\})$ and $q = P(\{b\})$, the probability that a occurs before b. 15

2. Let $(\Omega, \mathcal{P}(\Omega), P)$ be a discrete probability space with $P(\{a\}) > 0$ for all $a \in \Omega$. Define $d(A, B) = P(A \Delta B)$ for $A, B \in \mathcal{P}(\Omega)$. Show that d is a metric on $\mathcal{P}(\Omega)$. × 15

3. Suppose it is given in a probability space that at least one, but no more than three, of the events A_r, $1 \leq r \leq n$, occur, where $n \geq 3$; the probability of at least two occurring is 1/2. Further if $P(A_r) = p$, $P(A_r \cap A_s) = q$, $r \neq s$ and $P(A_r \cap A_s \cap A_t) = x$, $r < s < t$, then show that $p \geq \frac{3}{2n}$ and $q \leq \frac{4}{n}$. 20

   $np - \binom{n}{2}q + \binom{n}{3}x = 1$

4. Let X and Y be discrete random variables with mean 0, variance 1 and covariance c. Prove that $E(Max\{X^2, Y^2\}) \leq 1 + \sqrt{1 - c^2}$. 20

5. Define the conditional variance of Y given X by $Var(Y/X) = E\left((Y - E(Y/X))^2 / X\right)$. Show that $Var(Y) = E(Var(Y/X)) + Var(E(Y/X))$. 20

   $E(Y^2|X) - 2Y$

6. Let X and Y be independent random variables each having a geometric density with parameter p. Find $E(Y/X + Y)$. 20

$(E(Y))^2 -$ $P(X=1) =$

$\boxed{2}$ 02 11 20 $\to p^2(1-p)^2$

$p(1-p)^2$ $p^2(1-p)^2$ $p^2(1-p)^2$

$E(E(Y - E(Y|X))^2 | X)$

$E((Y - E(Y|X))^2))$

$E(Y^2|X)$ 03 $3 - 1$

$P(Y | X+Y)$

$+ (E(Y|X))^2$

$-2E(Y E(Y|X)|X)$ $\frac{P(X+Y|Y) \cdot P(Y)}{P(X+Y)}$ $E(Y^2|X)$

$+ E((E(Y|X))^2 |X)$ $E(Y^2) - 2$

$E(2Y E(Y|X))$

$E(Y|X)$

$-2E(Y f(x))$

$E(2Y\ E(Y|X))$

$2f(x)E(y)$ $E((f(x))^2)$ $(f(x))^2$ $2\ E(Y E(Y|X))$

$2Y\ E(E(Y|X))$