Quiz 1 - Probability Theory

7th February, 2025

Instructions

Solve all questions.
You are not obliged to use any of the hints. Any correct solution will be accepted.
RELAX. Think.

Problem 1

1. Let $\mathcal{F}$ be a σ-algebra. If $|\mathcal{F}| < \infty$, then prove that $|\mathcal{F}| = 2^n$ for some natural number $n \geqslant 1$. (Hint: Let G be a finite group. Suppose $g^2 = 1, \forall g \in G$. Use the following fact: if prime $p|\mathrm{ord}(G)$ then $\exists x \in G$ such that $\mathrm{ord}(x) = p$.)

Problem 2

1. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space such that $\mathbb{P}(A) \in \{0,1\}, \forall A \in \mathcal{F}$. Let $X : \Omega \to \mathbb{R}$ be a random variable. (Note: X need not be discrete.) Prove that there exists $c \in \mathbb{R}$ such that $\mathbb{P}(X = c) = 1$.

Problem 3

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space. Let $\{A_n\}$ be a sequence of events in $\mathcal{A}$. We define:

$$\limsup_n A_n = \{\omega : \omega \in A_n \text{ for infinitely many } n \in \mathbb{N}\}$$

$$\liminf_n A_n = \{\omega : \exists m \in \mathbb{N} \text{ such that } \omega \in A_n, \forall n \geqslant m\}$$

1. (*Fatou's lemma*) Prove that

$$\mathbb{P}(\liminf A_n) \leqslant \liminf \mathbb{P}(A_n) \leqslant \limsup \mathbb{P}(A_n) \leqslant \mathbb{P}(\limsup A_n)$$

2. (*Borel-Cantelli lemma*) Let $\{B_n : n \in \mathbb{N}\}$ be a sequence of events such that $\sum_{n\in\mathbb{N}} \mathbb{P}(B_n) < \infty$. Prove that $\mathbb{P}(\limsup_n B_n) = 0$.

Hint: Rewrite $\limsup_n A_n$ and $\liminf_n A_n$ using intersections and unions.