

# Quiz 1 - Probability Theory

$$ii = i \cdot i$$

$$\begin{aligned} ij &= i \cdot j = i^j = j^i = ji \\ ij &= i^j = j^i = ji \end{aligned}$$

*R grant*

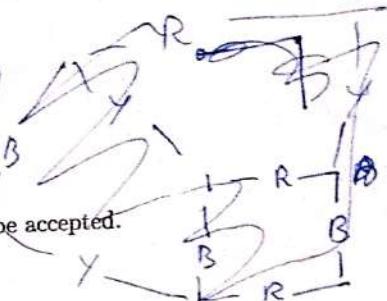
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7th February, 2025

$$P(A)P(A)$$

$$P(A \cup B) = P(A) + P(B)$$

$$P(A)P(B) = P(A \cap B)$$



## Instructions

Solve all questions.

You are not obliged to use any of the hints. Any correct solution will be accepted.  
RELAX. Think.

## Problem 1

1. Let  $\mathcal{F}$  be a  $\sigma$ -algebra. If  $|\mathcal{F}| < \infty$ , then prove that  $|\mathcal{F}| = 2^n$  for some natural number  $n \geq 1$ . (Hint: Let  $G$  be a finite group. Suppose  $g^2 = 1, \forall g \in G$ . Use the following fact: if prime  $p \mid \text{ord}(G)$  then  $\exists x \in G$  such that  $\text{ord}(x) = p$ .)

## Problem 2

1. Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space such that  $\mathbb{P}(A) \in \{0, 1\}, \forall A \in \mathcal{F}$ . Let  $X : \Omega \rightarrow \mathbb{R}$  be a random variable. (Note:  $X$  need not be discrete.) Prove that there exists  $c \in \mathbb{R}$  such that  $\mathbb{P}(X = c) = 1$ .

## Problem 3

Let  $(\Omega, \mathcal{A}, \mathbb{P})$  be a probability space. Let  $\{A_n\}$  be a sequence of events in  $\mathcal{A}$ . We define:

$$\limsup_{n \rightarrow \infty} A_n = \limsup_{n \rightarrow \infty} \{ \omega : \omega \in A_n \text{ for infinitely many } n \in \mathbb{N} \}$$

$$\liminf_{n \rightarrow \infty} A_n = \liminf_{n \rightarrow \infty} \{ \omega : \exists m \in \mathbb{N} \text{ such that } \omega \in A_n, \forall n \geq m \}$$

1. (Fatou's lemma) Prove that

$$\mathbb{P}(\liminf_{n \rightarrow \infty} A_n) \leq \liminf_{n \rightarrow \infty} \mathbb{P}(A_n) \leq \limsup_{n \rightarrow \infty} \mathbb{P}(A_n) \leq \mathbb{P}(\limsup_{n \rightarrow \infty} A_n)$$

$$\lim_{n \rightarrow \infty} \mathbb{P}(A_n) \quad \text{using } \lim_{n \rightarrow \infty} \mathbb{P}(A_n)$$

$$\leq \lim_{n \rightarrow \infty} \mathbb{P}(A_n) \quad \text{if } n = \infty$$

□

2. (Borel-Cantelli lemma) Let  $\{B_n : n \in \mathbb{N}\}$  be a sequence of events such that  $\sum_{n \in \mathbb{N}} \mathbb{P}(B_n) < \infty$ . Prove that  $\mathbb{P}(\limsup_n B_n) = 0$ .

Hint: Rewrite  $\limsup_n A_n$  and  $\liminf_n A_n$  using intersections and unions.

$$\begin{aligned} & P\left(\bigcup_{k=n}^{\infty} A_k\right) \\ &= P\left(\bigcup_{k=1}^{\infty} A_k\right) \quad \text{(K is. wt)} \\ & \quad \text{if } i \neq j = \lim_{n \rightarrow \infty} \bigcup_{k=n}^{\infty} A_k = 0 \\ & \quad \text{if } i = j = \lim_{n \rightarrow \infty} \bigcup_{k=n}^{\infty} A_k = 1 \end{aligned}$$

$$ij = i^{-1} j^{-1} \lim_{n \rightarrow \infty} \bigcup_{k=n}^{\infty} A_k = \lim_{n \rightarrow \infty} \bigcup_{k=n}^{\infty} A_k = 1 \quad \text{if } i = j$$

$$ij = i \cdot j \cdot i \cdot j$$

$$\begin{aligned} & P(A) = \lim_{n \rightarrow \infty} \bigcup_{k=n}^{\infty} A_k \\ & \quad \text{if } i \neq j = \lim_{n \rightarrow \infty} \bigcup_{k=n}^{\infty} A_k = 0 \\ & \quad \text{if } i = j = \lim_{n \rightarrow \infty} \bigcup_{k=n}^{\infty} A_k = 1 \end{aligned}$$

