# Probability Theory

## Quiz 3

Februrary 14, 2025

### Instructions

- Attempt all 5 problems. The total marks will be capped at 35 (subject to change).
- Begin each solution on a new sheet of paper.
- **ReLaX** (but not too much—time is limited, and we don't want you napping mid-quiz!) and **THINK** (but don't overthink— we've designed the questions to be easy, or at least broken them down into manageable parts!).

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### Problem 1 $\boxed{5}$

(a) Two subsets $X$ and $Y$ are chosen independently and uniformly at random from the power set of $[n] = \{1, 2, \ldots, n\}$. Show that:

$$\Pr(X \subseteq Y) = \Pr(X^c \cap Y^c = \emptyset).$$

**Note:** What is the intuitive reason that these two probabilities are equal? Think! *2*

(b) Suppose we repeatedly toss a biased coin until we get a head, where the probability of obtaining a head in each toss is $p$, independently of previous tosses. Let $T$ be the number of tosses required to obtain the first head. We define a random variable $\mathcal{N}$ as the number of trailing zeros in $T!$. *3*

(i) Show that $\Pr(\mathcal{N} = 17) = 0$. **Hint:** $\lfloor x \rfloor \leq x$.

(ii) Compute $\Pr(5 \leq \mathcal{N} \leq 30)$. **Hint:** First determine the number of trailing zeros in $T!$ for values of $T$ that are powers of *something*.

$(1-p)^{24}\left(1-(1-p)^{100}\right)$

### Problem 2 $\boxed{7}$

(a) We want to prove this identity, using basic probability theory:

$$\sum_{k=0}^{n} \binom{2k}{k} \binom{2n-2k}{n-k} = 2^{2n}$$

Start with an urn containing one red marble and one blue marble. Make $n$ draws from the urn; for each draw, remove a random ball in the urn, then put it back, along with two extra balls of the same color. What is the probability that exactly $k$ of the draws were red? Now sum over $k$.
**Hint:** You may use the notation $m!! = \prod_{j=0}^{\lfloor \frac{m}{2} \rfloor - 1} (m - 2j) = m(m-2)(m-4)\ldots$ *3*

(b) Consider a train with $n$ compartments and $m > n$ passengers. Suppose passengers board compartments uniformly at random. What is the probability that at least one person is in each compartment? Mention your answer in the case that $n = 4, m = 6$. *4*

### Problem 3 $\boxed{11}$

Let $S_i^n$ be the collection of all subsets of $[n] = \{1, 2, \ldots, n\}$ of size $i$. A subset is chosen uniformly at random from $S_i^n$, and we define the random variables: $\mathcal{A}_i^n, \mathcal{B}_i^n, \mathcal{M}_i^n, \mu_i^n : S_i^n \to \mathbb{R}$, where $\mathcal{A}_i^n$ is the minimum, $\mathcal{B}_i^n$ is the maximum, $\mathcal{M}_i^n$ is the median (for even $i$, the lower median is taken), and $\mu_i^n$ is the mean of a randomly selected subset of size $i$ from $[n]$. For parts (b), (d), (e), (f), assume $n = 5$ and $i = 3$ unless otherwise specified. Answer the following:

(a) Compute $\Pr(\mathcal{B}_{20}^{25} = \mathcal{M}_{20}^{25})$. *0.5*

$(1-p)^{24}\left(1-(1-p)^{100}\right)$



$24+4=28$

$125$

1 2 3 4 5 124 235
125

(b) Determine the conditional probability $\Pr(\mathcal{A} = i \mid \mathcal{M} = j)$, for integers $i$ and $j$ within their respective supports (i.e., the set of values each variable can take with nonzero probability). Comment on whether $\mathcal{M}$ and $\mathcal{B}$ are independent. **3**

$\frac{1}{j-1}$

(c) Find the smallest interval $[a, b]$ that contains the support of $\mu_3^n$ for $n \geq 3$. **0.5** $[2, n-1]$

(d) Find the probability that $\mu = \mathcal{M}$. **3** $\rightarrow \frac{2}{5}$

$\sum i \frac{(i-1)(5-i)}{10} = \frac{2\times1\times3 + 3\times2\times2 + 4\times3\times1}{10} = 3$

(e) Compute the expected values $\mathbb{E}(\mathcal{A})$, $\mathbb{E}(\mathcal{B})$, and $\mathbb{E}(\mathcal{M})$. **3** 3 $\frac{3}{2}$ $\frac{9}{2}$

(f) Derive a linear relation among the random variables $\mu_i^n, \mathcal{A}_i^n, \mathcal{B}_i^n$, and $\mathcal{M}_i^n$. Recall that if $X = \sum_{p=1}^{q} X_p$, then by linearity of expectation, $\mathbb{E}(X) = \sum_{p=1}^{q} \mathbb{E}(X_p)$. Using this relation, compute $\mathbb{E}(\mu_3^5)$. **2** $\frac{5}{2}$

**Note:** After the quiz, compute the probability that $\mathcal{M}_3^7 = \mathcal{B}_3^7 - \mathcal{A}_3^7$, meaning that the median of the set equals the range of the set.

## Problem 4

**10**

A drunk man is standing at the edge of a cliff. One step forward will send him over the edge, to his death. He steps randomly backward (with probability $p$) or forward (with probability $q = 1 - p$) at each time instant $t = 1, 2, 3, \ldots$. What are his chances of escaping alive?Follow these steps (if you want to):

(a) Think of our system as the motion of a particle along $\mathbb{N} \cup \{0\}$. The particle starts at $x = 1$ and can move forward to $x = 0$ w.p. $1 - p$ (where it will be absorbed), or backward w.p. $p$ to $x = 2$. Generalize this to when the particle is at $x = n$ at some stage and moves either forward or backward. **1**

(b) Call $P_k$ the probability of starting at $x = k$ and eventually being absorbed at $x = 0$. In terms of this, what do we wish to find? **1**

(c) Try to find an expression relating $P_1$ and $P_2$. **3**

(d) Now, try to express $P_2$ in terms of $P_1$ and $p$. Plug this into your expression from (c). **2**

(e) You might have ended up with multiple possible solutions to your expression. Try to argue for which values of $p \in [0, 1]$, which solutions hold. Hint: Do you expect (no need to prove) $P_1$ as a function of $p$ to have any "special" properties? **3**

## Problem 5

**7**

You want to send your partner Valentine's gifts. However, their overprotective father is always on the lookout, making it extremely risky for you. Every night, you try to sneak into their garden and keep the gift under the window. If their father catches you, you are banned from seeing your partner forever!

Each night, the probability that you are **not caught** by the father on the $(n+1)$th night, given that you have successfully avoided detection (i.e., not caught) and delivered the gift on all previous nights, is given by $v(n) > 0$. Let $X$ denote the number of nights you successfully deliver the gifts before finally getting caught.

(a) Find the probability that $X = n$ by following these steps. Assume that $E_n$ is the event of not getting caught on night $n$ (note that the given probability $v(n)$ refers to a different event): $V(n-1) \prod_{i=0} (1 - V(i))$

(i) Express the event that $X = n$ in terms of $E_i$ for an appropriate range of $i$ using set-theoretic operations such as union, intersection, and complements. **1**

(ii) Recall that $P(B \cap A)$ can be written as $P(B \mid A)P(A)$. Using this fact recursively, determine the value of $\Pr(X = n)$ in terms of the function $v$. **2**

(b) Prove that for any random variable taking only non-negative integral values,

$$\mathbb{E}(X) = \sum_{x=1}^{\infty} \Pr(X \geq x)$$

**1**

(c) Solve **EITHER** subpart (i) **OR** (ii): **3**

(i) Suppose $v(n) = \frac{n+1}{n+2}$. Compute $P(X = n)$ explicitly in terms of $n$. What can you infer about the expected value of $X$?

(ii) Suppose $v(n) = 3^{-\lambda(n+1)}$ for some fixed positive real $\lambda$. Compute $P(X = n)$ explicitly in terms of $n$. What can you infer about the expected value of $X$?

**Note:** After completing the quiz, reflect on the meaning of the functions in the last two subparts and why the expected value behaves as it does. (Also, consider the moral of the story: *some things* require persistence, but sometimes you must calculate the risks carefully!)

$n - 2 - n + 2\,3$ $\quad -\lambda n$

$n + 2 - n - \frac{1}{n+2}$ $\quad \frac{1}{n+2}$ $\quad 3$ $\quad (1-$