Quiz 3

11th April, 2025

Instructions

There is only one question, broken down into six parts.
There is a bonus question that you can attempt for extra credit.
Do NOT hesitate to ask me if you find the notation confusing.
RELAX. Think.

Problem

Given a discrete random variable $X : \Omega \to S$, the measure μ_X induced by X is defined as:

$$\mu_X(A) = \mathbb{P}(X \in A),\ \forall A \subseteq S$$

Let $X_{n,m}$, $1 \leq m \leq n$ be a triangular array of random variables such that

$$\mathbb{P}(X_{n,m} = 1) = p_{n,m} \text{ and } \mathbb{P}(X_{n,m} = 0) = (1 - p_{n,m})$$

$f_{X_{n,m}} = p_{n,m} e^{it} + 1 - p_{n,m}$

We have the following conditions:

$$\lim_{n \to \infty} \sum_m p_{n,m} = \lambda \in (0, \infty) \text{ and } \lim_{n \to \infty} \max_{1 \leq m \leq n} p_{n,m} = 0$$

Let $S_n = (X_{n,1} + \ldots X_{n,n})$. We want to show that

$$S_n \Rightarrow \text{Poisson}(\lambda) \tag{†}$$

Let μ and ν be probability measures on $(\Omega, 2^\Omega)$, where Ω is coutable. Define:

$$\|\mu - \nu\| \equiv \frac{1}{2} \sum_z |\mu(z) - \nu(z)| = \sup_{A \subseteq \Omega} |\mu(A) - \nu(A)| \tag{††}$$

The first part of this problem provides a guided proof of (†), while the second part, for ***bonus marks****, asks you to supply an alternative proof of the same result.*

1. Follow the given outline:
   (a) Let Z_n, $1 \leq n \leq \infty$, be integer valued. Show that $Z_n \Rightarrow Z_\infty$ if and only if $P(Z_n = m) \to P(Z_\infty = m)$ for all $m \in \mathbb{Z}$.
   (b) Notice that $d(\mu, \nu) = \|\mu - \nu\|$ defines a metric on the set of probability measures on $\mathbb{Z}$. Show that $\|\mu_n - \mu\| \to 0$ if and only if $\mu_n \Rightarrow \mu$. (Hint: How would you prove the second equality in (††)? For which A is the maximum attained?)
   (c) If $\mu_1 \times \mu_2$ denotes the product measure on $\mathbb{Z} \times \mathbb{Z}$ that has $(\mu_1 \times \mu_2)(x, y) = \mu_1(x)\mu_2(y)$, then show that
   $$\|\mu_1 \times \mu_2 - \nu_1 \times \nu_2\| \leq \|\mu_1 - \nu_1\| + \|\mu_2 - \nu_2\|$$
   (d) If $\mu_1 * \mu_2$ denotes the convolution of μ_1 and μ_2, that is, $\mu_1 * \mu_2(x) = \sum_y \mu_1(x - y)\mu_2(y)$, then show that $\|\mu_1 * \mu_2 - \nu_1 * \nu_2\| \leq \|\mu_1 \times \mu_2 - \nu_1 \times \nu_2\|$.

$|\mu_1(z)\mu_2(z) - \nu_1(z)\nu_2(z)|$

$< |\mu_1(z) - \nu_1(z)| + |\mu_2(z) - \nu_2(z)|$

$1 - p e^{-p}$ $\qquad \frac{\lambda^x e^{-\lambda}}{x!}$

$< 1 - e^{-p}$

(e) Let μ be the measure with $\mu(1) = p$ and $\mu(0) = 1 - p$. Let ν be a Poisson distribution with mean p. Then show

$$\|\mu - \nu\| \leq p^2$$

(Hint: $1 - x \leq e^{-x}$.)

(f) Now prove (†). (Hint: What distribution does the sum of two independent Poisson random variables follow?)

2. (*Bonus*) Prove (†) using characteristic functions. (Hint: Let $z_1, \ldots, z_n$ and $w_1, \ldots, w_n$ be complex numbers of modulus ≤ 1. Then $|\prod_{m=1}^n z_m - \prod_{m=1}^n w_m| \leq \sum_{m=1}^n |z_m - w_m|$.)

$1 > e^{-p}$ $\qquad e^{p} > 1$ $\qquad e^{-p} + p - 1 < p^2$

$e^{p} < 1$

$\frac{1}{2}(e^{-p} - p) < p^2$ $\qquad 1 - x < 1 - x + \frac{x^2}{2} - \ldots$

$e^{-p} - p < 2p^2$ $\qquad \frac{x^3}{3} - \frac{x^4}{4} \ldots < \frac{x^2}{2}$

$e^{-p} < p(2p+1)$

$\frac{1}{p(2p+1)} < e^{p}$ $\qquad 1 - x \leq 1 - x + \frac{x^2}{2} - \frac{x^3}{6}$ $\qquad 1 + x + \frac{x^2}{2} + \ldots$

$1 + x + x \frac{x^3}{3}$

$1 - e^{-p} - x$

$e^{-p} + x - 1 < p^2$

$0 < \frac{x^2}{2} - \ldots$ $\qquad 1 - p + \frac{p^2}{2} - \frac{p^3}{6} + \ldots$

$\frac{x^3}{6} < \frac{x^2}{2}$ $\qquad \frac{p^2}{2} - \frac{p^3}{6} + \ldots$

$\frac{p^2}{2}\left(1 - \frac{p}{3} + \ldots\right)$

$\frac{p^2}{2}\left(1 + \frac{p}{3} + \ldots\right)$

$(1 + p + \frac{p^2}{2} + \ldots)$

$< \frac{p^2}{2} e^{p}$ $\qquad \frac{p^2}{2} < 2$

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