Probability Theory

Quiz 5

April 16, 2025

$\frac{z}{1-z} + 1$

$z \frac{1}{1-z}$

Instructions

- The quiz carries a total of 20 marks. The time for completion is 90 minutes.
- Attempt all 3 problems.
- Begin the solution to each problem on a new sheet of paper.
- You may use any facts stated in the **Appendix** while writing your solutions.
- During the quiz, direct any questions related to Problems 1 and 2 to **Vardhan**, and questions related to Problem 3 to **Harini**.

Problem 1 $\boxed{5}$

Let $X \sim \text{Gamma}(\alpha, \lambda)$ and $Y \sim \text{Gamma}(\beta, \lambda)$ be independent random variables, where $\alpha, \beta, \lambda > 0$. (See the Appendix for definitions.)

1. If $Z = \frac{X}{X+Y}$, show that

$$\frac{\alpha}{\alpha+\beta} \qquad \mathbb{E}(Z) = \frac{\mathbb{E}(X)}{\mathbb{E}(X) + \mathbb{E}(Y)}.$$

$z = \frac{X}{X+Y}$

$X + Y = \frac{1}{z}(X)$

$Y = \left(\frac{1}{z} - 1\right)X$

2. Find the value of $\Gamma\left(\frac{3}{2}\right)$.
(Hint: You may use the facts given in the appendix.)

$E(Z) = E\left(\frac{X}{X+Y}\right)$

Problem 2 $\boxed{5}$

Suppose there are r rooms in a hostel, and let R_i denote the event that the ith room is occupied (these events need not be independent). Let N be the number of rooms that are occupied. For $s \in [r] := \{1, 2, \ldots, r\}$, let $\mathscr{F}(s)$ denote the collection of all strictly increasing functions from $[s]$ to $[r]$; that is,

$$\mathscr{F}(s) = \{f : [s] \to [r] \mid f(1) < f(2) < \cdots < f(s)\}.$$

Define:

$$\alpha(s) = \sum_{f \in \mathscr{F}(s)} \Pr\left(\bigcap_{i=1}^{s} R_{f(i)}\right) \quad \text{and} \quad G_\alpha(z) = \sum_{s=0}^{r} \alpha(s) z^s, \text{ take } \alpha(0) = 1$$

Show that

$$\Pr(N = k) = \sum_{s=k}^{r} (-1)^{s-k} \alpha(s) \binom{s}{k}.$$

(Hint: First show that $\alpha(s) = \mathbb{E}\left[\binom{N}{s}\right]$, then expand $G_\alpha(z-1)$.)

(Optional: Try this after the quiz. This part will not be graded.)

Show that

$$\alpha(s) = \sum_{i=s-1}^{r-1} \binom{i}{s-1} (1 - F_N(i)),$$

where F_N denotes the cumulative distribution function (CDF) of N.

$\int \int n^2 \frac{1}{n} dn \quad \int_0^n \frac{1}{n} x^2 dx$

$\frac{1}{n}\left(\frac{x^3}{3}\right) \quad \frac{n^2}{\pi} \quad \frac{1}{3}$

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Problem 3

Let (X_n, Z_n) be a sequence of random variables on $\mathbb{R}^2$ such that $X_n \sim \text{Unif}[-n, n]$, $Y \sim N(0,1)$, independent of X_n, and $Z_n = X_n^2 + Y$ on $\mathbb{R}$.

1. (a) Write down the probability density function (pdf) of X_n as $f_{X_n}(x)$. *1*
   (b) Compute the joint density $f_{X_n,Z_n}(x,z)$. (Hint: use $Z_n \mid X_n$.) *1.5*
2. (a) Compute $\mathbb{E}[Z_n \mid X_n = x]$ and $\text{Var}(Z_n \mid X_n = x)$. *1*
   (b) Compute $\mathbb{E}[Z_n]$ and $\text{Var}(Z_n)$,
   i. without using (a) or (b) *1*
   ii. using (a) and properties of conditional expectation and variance from the appendix. *1*
3. (a) Does the sequence (X_n, Z_n) converge in distribution as $n \to \infty$? Check this using only the definition of convergence in distribution from the Appendix. *1*
   (b) State the continuity theorem for characteristic functions in 1D. Extend this to the 2D case we are considering, using the definition of the joint characteristic function above. *1*
4. (a) Compute the joint characteristic function
   $$\phi_{X_n,Z_n}(s,t) := \mathbb{E}[e^{isX_n + itZ_n}],$$
   and express it as an integral of a function in one variable (Hint: use $Z_n = X_n^2 + Y$). *1.5*
   (b) Levy's Continuity Theorem states that $(X_n, Z_n) \Rightarrow (X, Z)$ for some pair of random variables X and Z on $\mathbb{R}$ $\iff \phi_{X_n,Z_n}(s,t) \to \phi_{X,Z}(s,t)\ \forall (s,t) \in \mathbb{R}^2$.
   Does $\phi_{X_n,Z_n}(s,t)$ converge pointwise? Use this to verify your answer in 3(a). *1*

Appendix

1. If $A \sim \text{Gamma}(a, b)$, then
$$f_A(x) = \begin{cases} \frac{1}{\Gamma(a)} b^a x^{a-1} e^{-bx} & \text{if } x > 0, \\ 0 & \text{otherwise.} \end{cases}$$
Also, $\mathbb{E}(A) = \frac{a}{b}$ and $\text{Var}(A) = \frac{a}{b^2}$.
2. The function $\Gamma(a)$ is defined as
$$\Gamma(a) = \int_0^\infty x^{a-1} e^{-x} dx.$$
It satisfies the identity:
$$\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} = \int_0^1 x^{a-1}(1-x)^{b-1} dx.$$
For $a \in \mathbb{Z}^+$, $\tau(a) = (a-1)!$.
3.
$$\int_{-1/2}^{1/2} \left(\frac{1}{2} + x\right)^{1/2} \left(\frac{1}{2} - x\right)^{1/2} dx = \frac{\pi}{8}.$$
4. Let $(Xn : n \geq 1)$ and X be random variables. Denote the corresponding distribution functions by F_n and F respectively. We say that $X_n \implies X$ (or X_n converges in distribution to X) $\iff \forall$ continuous bounded functions f, $\mathbb{E}[f(X_n)] \to \mathbb{E}[f(X)]$.
5. Given 2 random variables X and Y on $\mathbb{R}$, $\mathbb{E}[Y \mid X]$ is a random variable $x \mapsto \mathbb{E}[Z_n \mid X_n = x]$. Taking its expectation gives a number $\mathbb{E}[\mathbb{E}[Y \mid X]]$.
   - Law of total expectation: $\mathbb{E}[\mathbb{E}[Y \mid X]] = \mathbb{E}[Y]$.
   - Law of total variance: $Var(Y) = \mathbb{E}[Var(Y|X)] + Var(\mathbb{E}[Y|X])$

$x = \frac{1}{2} + u \qquad \frac{z}{1-z} + 1 \qquad \frac{1-z}{z} + 1 \qquad \frac{1}{z}$

$\frac{(\Gamma(\frac{3}{2}))^2}{\Gamma(3)} \qquad \frac{y}{1-z} = u^{1-z}$

$\frac{\Gamma(\frac{3}{2})^2}{2} = \int_0^1 x^{1/2}(1-x)^{1/2} dx \implies \frac{1}{1-z} dy = du$

$(1-z)^{1-\alpha} \qquad 1 - \alpha + \alpha + \beta - z + 1$