# **Probability Theory**

# Quiz 5

April 16, 2025

#### Instructions

- The quiz carries a total of 20 marks. The time for completion is 90 minutes.
- Attempt all 3 problems.
- · Begin the solution to each problem on a new sheet of paper.
- · You may use any facts stated in the Appendix while writing your solutions.
- · During the quiz, direct any questions related to Problems 1 and 2 to Vardhan, and questions related to Problem 3 to Harini.

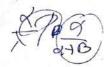




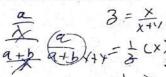
#### Problem 1

Let  $X \sim \text{Gamma}(\alpha, \lambda)$  and  $Y \sim \text{Gamma}(\beta, \lambda)$  be independent random variables, where  $\alpha, \beta, \lambda > 0$ . (See the Appendix for definitions.)

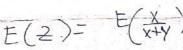
1. If 
$$Z = \frac{X}{X+Y}$$
, show that



$$\mathbb{E}(Z) = \frac{\mathbb{E}(X)}{\mathbb{E}(X) + \mathbb{E}(Y)}$$



2. Find the value of  $\Gamma(\frac{3}{2})$ . (Hint: You may use the facts given in the appendix.)



### Problem 2

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Suppose there are r rooms in a hostel, and let Ri denote the event that the ith room is occupied (these events need not be independent). Let N be the number of rooms that are occupied. For  $s \in [r] := \{1, 2, ..., r\}$ , let  $\mathcal{F}(s)$  denote the collection of all strictly increasing functions from [s] to [r]; that is,

$$\mathcal{F}(s) = \{ f : [s] \to [r] \mid f(1) < f(2) < \dots < f(s) \} \,.$$

Define:

$$\alpha(s) = \sum_{f \in \mathcal{F}(s)} \Pr\left(\bigcap_{i=1}^{s} R_{f(i)}\right)$$
 and  $G_{\alpha}(z) = \sum_{s=0}^{r} \alpha(s) z^{s}$ , take  $\alpha(0) = 1$ 

Show that

$$P_{\mathbb{F}}(N=k) = \sum_{s=k}^{r} (-1)^{s-k} \alpha(s) \binom{s}{k}.$$

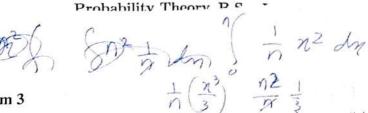
(Hint: First show that  $\alpha(s) = \mathbb{E}\left[\binom{N}{s}\right]$ , then expand  $G_{\alpha}(z-1)$ .)

(Optional: Try this after the quiz. This part will not be graded.)

Show that

$$\alpha(s) = \sum_{i=s-1}^{r-1} \binom{i}{s-1} (1 - F_N(i)),$$

where  $F_N$  denotes the cumulative distribution function (CDF) of N.



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Problem 3

Let  $(X_n, Z_n)$  be a sequence of random variables on  $\mathbb{R}^2$  such that  $X_n \sim \text{Unif}[-n, n]$ ,  $Y \sim \mathcal{N}(0, 1)$ , independent of  $X_n$ , and  $Z_n = X_n^2 + Y$  on  $\mathbb{R}$ .

- 1 1.5 1. (a) Write down the probability density function (pdf) of  $X_n$  as  $f_{X_n}(x)$ .
  - (b) Compute the joint density  $f_{X_n,Z_n}(x,z)$ . (Hint: use  $Z_n \mid X_n$ .)
- 2. (a) Compute  $\mathbb{E}[Z_n \mid X_n = x]$  and  $\text{Var}(Z_n \mid X_n = x)$ .
  - (b) Compute E[Z<sub>n</sub>] and Var(Z<sub>n</sub>), i. without using (a) or (b)
    - ii. using (a) and properties of conditional expectation and variance from the appendix.
- (a) Does the sequence  $(X_n, Z_n)$  converge in distribution as  $n \to \infty$ ? Check this using only the definition of convergence in distribution from the Appendix.
  - (b) State the continuity theorem for characteristic functions in 1D. Extend this to the 2D case we are considering, I using the definition of the joint characteristic function above.
- 4. (a) Compute the joint characteristic function

$$\phi_{X_n,Z_n}(s,t) := \mathbb{E}[e^{isX_n + itZ_n}],$$

1.5 and express it as an integral of a function in one variable (Hint: use  $Z_n = X_n^2 + Y$ ).

(b) Levy's Continuity Theorem states that  $(X_n, Z_n) \Rightarrow (X, Z)$  for some pair of random variables X and Z on R  $\iff \phi_{X_n,Z_n}(s,t) \to \phi_{X,Z}(s,t) \ \forall (s,t) \in \mathbb{R}^2.$ 

Does  $\phi_{X_n,Z_n}(s,t)$  converge pointwise? Use this to verify your answer in 3(a).

## Appendix

1. If  $A \sim \text{Gamma}(a, b)$ , then

$$f_A(x) = \begin{cases} \frac{1}{\Gamma(a)} b^a x^{a-1} e^{-bx} & \text{if } x > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Also,  $E(A) = \frac{a}{b}$  and  $Var(A) = \frac{a}{b^2}$ .

2. The function  $\Gamma(a)$  is defined as

$$\Gamma(a) = \int_0^\infty x^{a-1} e^{-x} dx.$$

It satisfies the identity:

$$\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}=\int_0^1 x^{a-1}(1-x)^{b-1}dx.$$

For  $a \in \mathbb{Z}^+$ ,  $\tau(a) = (a-1)!$ .

3. 
$$\int_{-1/2}^{1/2} \left(\frac{1}{2} + x\right)^{1/2} \left(\frac{1}{2} - x\right)^{1/2} dx = \frac{\pi}{8}.$$

- 4. Let  $(Xn : n \ge 1)$  and X be random variables. Denote the corresponding distribution functions by  $F_n$  and F respectively. We say that Let  $(An: n \ge 1)$  and X be fundamentally an expected by  $X_n \implies X$  (or  $X_n$  converges in distribution to X)  $\iff$  Y continuous bounded functions f,  $E[f(X_n)] \to E[f(X)]$ .
- 5. Given 2 random variables X and Y on R,  $E[Y \mid X]$  is a random variable  $x \mapsto E[Z_n \mid X_n = x]$ . Taking its expectation gives a number  $\mathbb{E}[\mathbb{E}[Y \mid X]].$ 
  - Law of total expectation: E[E[Y | X]] = E[Y].

