Theory of Computation

PSet-1

21 August 2025

This is a compilation of solutions for PSet-1 of Theory of Computation class instructed by Prof. C Aishwariya. The solutions are of original design.

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§0. Problems

1. Write a regular expression and draw an automaton for the following language over the alphabet $\Sigma = \{a, b, c\}$:

$$\{w \in \Sigma^* \mid |w|_a \geq 2 \Longleftrightarrow |w|_b \leq 2\}$$

   Here, $|w|_a$ refers to the number of occurrences of a in w, and similarly for $|w|_b$.

2. Let Σ be a (finite) alphabet. Given a language $L \subseteq \Sigma^*$, define $\mathrm{yolk}(L)$ as follows:

$$\mathrm{yolk}(L) := \{y \mid \exists x, z \in \Sigma^* \text{ s.t.} |x| = |y| = |z| \text{ and } xxyzz \in L\}.$$

   Show that if L is recognizable, then $\mathrm{yolk}(L)$ is also recognizable.

3. Let Σ be a (finite) alphabet. For words $x, y \in \Sigma^*$, we write $x \leq y$ if x is a prefix of y. For example, we have $ab \leq abcb$, but it is NOT the case that $ab \leq acb$.

   Prove that if L is a recognizable language, the set $\max(L)$ of maximal elements of L (with respect to $\leq$) is also recognizable. Prove also that if L is a recognizable language, then the set $\min(L)$ of minimal elements of L (with respect to $\leq$) is also recognizable.

§1. Problem 1

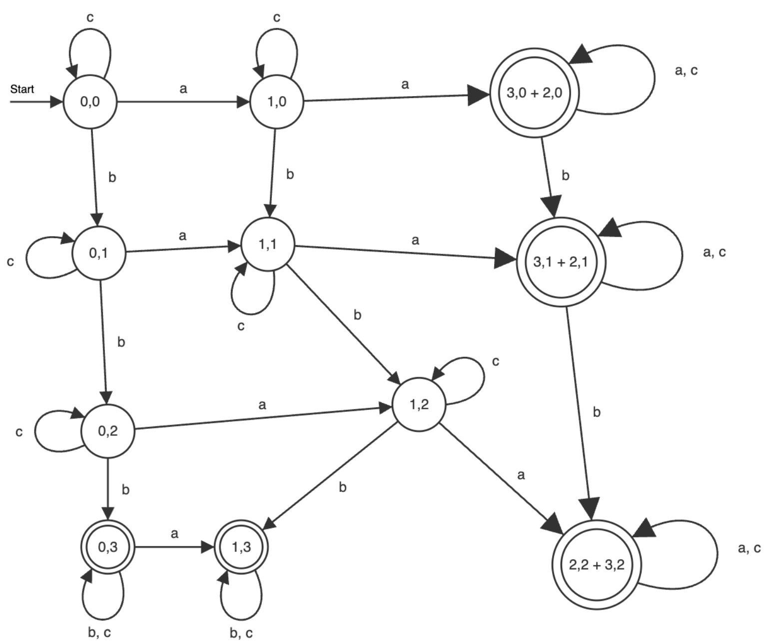
Exercise

Write a regular expression and draw an automaton for the following language over the alphabet $\Sigma = \{a, b, c\}$:

$$\{w \in \Sigma^* \mid |w|_a \geq 2 \iff |w|_b \leq 2\}$$

Here, $|w|_a$ refers to the number of occurrences of a in w, and similarly for $|w|_b$.

Solution.



The regex is

```
bbb
+ abbb + babb + bbab + bbba
+ aa
+ aab + aba + baa
+ aabb + abab + bbaa+ baba + abba + baab+
+ aaa
+ aaab + aaba + abaa + baaa +
+ aaabb + aabab + baaab + abaab + bbaaa + abbaa + babaa + ababa + baaba + aabba
```

□

§2. Problem 2

Exercise

Let Σ be a (finite) alphabet. Given a language $L \subseteq \Sigma^*$, define $\mathrm{yolk}(L)$ as follows:

$$\mathrm{yolk}(L) := \{y \mid \exists x, z \in \Sigma^* \text{ s.t.} |x| = |y| = |z| \text{ and } xxyzz \in L\}.$$

Show that if L is recognizable, then $\mathrm{yolk}(L)$ is also recognizable.

Proof. Let L be recognizable by $L = (Q, \Sigma, \delta, s, F)$.

We define $f_w = \hat{\delta}(q, w)$ for $w \in \Sigma^*$.

We claim that
- $Q' = \mathrm{pow}\left(Q^Q\right) \times Q^Q$
- $\Sigma' = \Sigma$
- $\delta'((W, f_w), a) = (\{f_{w' \cdot l} \mid f_{w'} \in W, l \in \Sigma\}, f_{w \cdot a})$
- $s' = (\{\mathrm{id}\}, \mathrm{id}) = (\{f_\varepsilon\}, f_\varepsilon)$
- $F' = \left\{\left(a, f_y\right) \mid \exists f_x, f_z \in a \text{ s.t. } \left(f_z \circ f_z \circ f_y \circ f_x \circ f_x\right)(s) \in F\right\}$

Lemma 3.1. *$\hat{\delta}'(s', y) = \left(a, f_y\right)$ where $a = \left\{f_w \mid w \in \Sigma^{|y|}\right\}$.*

Proof. We proceed via induction.

(B) If $|y| = 1$, then let $y = l \in \Sigma$. That implies,

$$\begin{aligned} & \hat{\delta}'(s, y) \\ &= \delta'(s, l) && \text{as } y = l \\ &= \delta((\{\mathrm{id}\}, \mathrm{id}), l) && \text{as } s = (\{\mathrm{id}\}, \mathrm{id}) \\ &= \delta((\{f_\varepsilon\}, f_\varepsilon), l) && \text{using definition} \\ &= (\{f_i \mid i \in \Sigma\}, f_l) && \text{using definition} \\ &= \left(\{f_w \mid w \in \Sigma^1\}, f_y\right) && \text{using } y = l \end{aligned}$$

(S) Assume that the claim holds for words of length $n - 1$, we will show that this implies that the claim holds for words of length n.

Let $|y| = n, y = w \cdot l$ where $w \in \Sigma^{|y| - 1}, l \in \Sigma$.

$$\begin{aligned} & \hat{\delta}'(s, y) \\ \iff & \hat{\delta}'(s, w \cdot l) && \text{using } y = w \cdot l \\ \iff & \delta'\left(\hat{\delta}'(s, w), l\right) && \text{by definition} \\ \iff & \delta'\left(\left(\left\{f_i \mid i \in \Sigma^{|y| - 1}\right\}, f_w\right), l\right) && \text{by induction hypothesis} \\ \iff & \left(\left\{f_{i \cdot j} \mid i \in \Sigma^{|y| - 1}, j \in \Sigma\right\}, f_{w \cdot l}\right) && \text{by definition} \\ \iff & \left(\left\{f_i \mid i \in \Sigma^{|y|}\right\}, f_y\right) && \text{by } |y| - 1 + 1 = |y| \text{ and } y = w \cdot l \end{aligned}$$

As required. □

With this, we will prove the correctness of this automata by showing the equivalence $y \in \mathrm{yolk}(L) \iff \hat{\delta}'(s', y) \in F'$.

$$\begin{aligned}
& y \in \mathrm{yolk}(L) & \\
\iff & \exists x, z \in \Sigma^* \text{ s.t.} |x| = |y| = |z|, xxyzz \in L & \text{by definition} \\
\iff & \exists x, z \in \Sigma^* \text{ s.t.} |x| = |y| = |z|, f_z \circ f_z \circ f_y \circ f_x \circ f_x(s) \in F & \text{by definition} \\
\iff & \exists x, z \in \Sigma^{|y|} \text{ s.t.} f_z \circ f_z \circ f_y \circ f_x \circ f_x(s) \in F & \text{as } |x| = |y| = |z| \\
\iff & \exists x, z \in a \text{ s.t.} f_z \circ f_z \circ f_y \circ f_x \circ f_x(s) \in F \text{ where } \left(a, f_y\right) = \hat{\delta}'(s', y) & \text{by lemma 3.1} \\
\iff & \hat{\delta}'(s', y) \in F' & \text{by definition}
\end{aligned}$$

And we are done! $\square$

§3. Problem 3

Exercise

Let Σ be a (finite) alphabet. For words $x, y \in \Sigma^*$, we write $x \leq y$ if x is a prefix of y. For example, we have $ab \leq abcb$, but it is NOT the case that $ab \leq acb$.

Prove that if L is a recognizable language, the set $\max(L)$ of maximal elements of L (with respect to $\leq$) is also recognizable. Prove also that if L is a recognizable language, then the set $\min(L)$ of minimal elements of L (with respect to $\leq$) is also recognizable.

Proof (Part 1). Given $\mathcal{A} = (Q, \Sigma, \delta, s, F)$ recognizes L, we claim that $\mathcal{A}' = (Q, \Sigma, \delta, s, \max(F))$ where $\max(F)$ is the set of maximal elements in the poset induced on F by the ordering $f_1 \leq f_2 \iff \exists w \in \Sigma^*$ s.t. $\hat{\delta}(f_1, w) = f_2$.

The correctness of the automata can be shown by showing the equivalence $x \in \max(L) \iff \hat{\delta}(s, x) \in \max(F)$

$$
\begin{aligned}
& x \in \max(L) \\
\iff & x \in L \land \nexists w \in \Sigma^* \text{ s.t.} xw \in L \\
\iff & \hat{\delta}(s, x) \in F \land \hat{\delta}(s, xw) \notin F \\
\iff & \hat{\delta}(s, x) \in F \land \hat{\delta}\left(\hat{\delta}(s, x), w\right) \notin F \\
\iff & \hat{\delta}(s, x) = f \in F \text{ s.t.} \hat{\delta}(f, w) \notin F \\
\iff & f \in \max(F) \\
\iff & \hat{\delta}(s, x) \in \max(F)
\end{aligned}
$$

□

Proof (Part 2). Given $\mathcal{A} = (Q, \Sigma, \delta, s, F)$ recognizes L, we claim $\mathcal{A}' = (Q \cup \{d\}, \Sigma, \delta', s, F)$ recognizes $\min(L)$ where:

$$
\delta'(q, k) = \begin{cases} \delta(q, k) \text{ if } q \notin F \\ \\ d \text{ if } q \in F \cup \{d\} \end{cases}
$$

.

The correctness of this autmoata can be shown by showing the equivalence $x \in \min(L) \iff \hat{\delta}(s, x) \in F$.

If there exists a prefix of x which is in L, we get moved to the trap state d. And if x is accepted by $\mathcal{A}'$, them we don't visit any node in F in the accepting run except at the end; thus, no prefix of x is in L.

Thus, x is accepted by $\mathcal{A}'$ if and only if none of it's prefix are in L that is if $x \in \min(L)$.

And we are done! □