Theory of Computation

PSet-2

06 September 2025

This is a compilation of solutions for PSet-2 of Theory of Computation class instructed by Prof. C Aishwariya. The solutions are of original design.

Table of contents

§0. Problems

1. Are the following subsets of $\{a, b, \$\}^*$ recognizable? Justify.

$$\{xy \mid x, y \in \{a, b\}^*, |x|_a = |y|_b\}$$

$$\{x\$y \mid x, y \in \{a, b\}^*, |x|_a = |y|_b\}$$

2. Let A be a DFA with n states. Is the following statement true? Substantiate.

   If A accepts all the words whose length is between n and $2n$, then $L(A) = \Sigma^*$. In other words, if $\{w \mid n \leq |w| \leq 2n\} \subseteq \Sigma^*$, then $L = \Sigma^*$.

3. Consider the grammar G below for if-then-else statements.

   $G = (\{S, C\}, \{i, t, e, c, s\}, P, S)$

   where the set of productions P is:

   $S \to iCtSeS \mid iCtS \mid s$

   $C \to c$

   (a) Argue that G is ambiguous.

   (b) Give an unambiguous grammar for $L(G)$?

§1. Problem 1

Exercise

Are the following subsets of $\{a, b, \$\}^*$ recognizable? Justify.

$$\{xy \mid x, y \in \{a, b\}^*, |x|_a = |y|_b\}$$

$$\{x\$y \mid x, y \in \{a, b\}^*, |x|_a = |y|_b\}$$

Proof. (a)

Claim 2.1. $\{a, b\}^* = \{xy \mid x, y \in \{a, b\}^*, |x|_a = |y|_b\}$.

Proof. It is trivial that $\{xy \mid x, y \in \{a, b\}^*, |x|_a = |y|_b\} \subseteq \{a, b\}^*$. We will prove the other side by induction.

Inducting on the length of strings

(B) For ε we can divide as $\varepsilon + \varepsilon$.

For strings of length 1, $a = \varepsilon + a$ and $b = b + \varepsilon$.

(S) Assuming such a division exists for strings of length n, we will show that such a division will exist for strings of length $n + 1$.

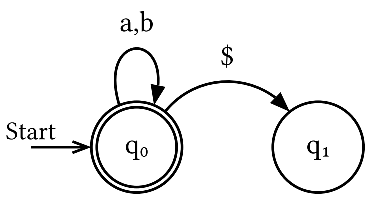
Case 1: If the new character is a, we can continue with the last division.

Case 2: If the new character is b, Consider
- $w_1 + \varepsilon + b$ is divided as $w_1 b + \varepsilon$ where $|w_1| = n, |w_1|_a = |\varepsilon|_b = 0 \Rightarrow w_1 = b^n$
- $w_1 + a w_2 + b$ is divided as $w_1 a + w_2 b$ where $|w_1 a w_2| = n, |w_1|_a = |a w_2|_b$
- $w_1 + b w_2 + b$ is divided as $w_1 b + w_2 b$ where $|w_1 b w_2| = n, |w_1|_a = |b w_2|_b$

Thus, $\{xy \mid x, y \in \{a, b\}^*, |x|_a = |y|_b\} \supseteq \{a, b\}^*$

Thus, by squeeze principle, $\{xy \mid x, y \in \{a, b\}^*, |x|_a = |y|_b\} = \{a, b\}^*$. □

As the sets are equal, we can use the same automata to recognize them.



Claim 2.2.

$$\{x\$y \mid x, y \in \{a, b\}^*, |x|_a = |y|_b\}$$

is unrecognizable.

Proof. We will use a game with demon (Kozen, equivalent to Pumping Lemma) to prove this.

For a given natural $k > 0$, let $x = \varepsilon, y = a^k, z = \$b^k$. This is valid as $a^k\$b^k$ is in the language.

Given demon's choice of u, v, w such that $uvw = y = a^k, v \neq \varepsilon \Rightarrow v = a^r, k \geq r > 0$.

We will choose $i = 2$. As $xuv^iwz = a^{k+r}\$b^k$ is not a part of language, we are done! $\square$

$\square$

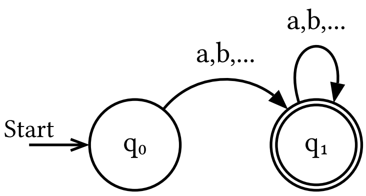
§2. Problem 2

Exercise

Let A be a DFA with n states. Is the following statement true? Substantiate.

If A accepts all the words whose length is between n and $2n$, then $L(A) = \Sigma^*$. In other words, if $\{w \mid n \leq |w| \leq 2n\} \subseteq \Sigma^*$, then $L = \Sigma^*$.

Proof. For $n = 2$, notice



Notice, ε is not accepted by this automata.

Furthermore, notice that $L(A) = \{w \mid |w| \geq 1\} \supset \{w \mid 2 \leq |w| \leq 4\}$.

Thus, $L(A) \neq \Sigma^*$, contradicting the given statement. □

§3. Problem 3

Exercise

Consider the grammar G below for if-then-else statements.

$G = (\{S, C\}, \{i, t, e, c, s\}, P, S)$

where the set of productions P is:

$S \to iCtSeS \mid iCtS \mid s$

$C \to c$

(a) Argue that G is ambiguous.

(b) Give an unambiguous grammar for $L(G)$.

Proof. (a) We claim $ictictses$ can be generated in atleast two ways.

$$\begin{aligned} S &\to iCtSeS &&\to icticCtSes \to ictictses \\ S &\to iCtS &&\to ictiCtSeS \to ictictses \end{aligned}$$

(b) We claim the following grammer works

$$G' = (\{S, C, S'\}, \{i, t, e, c, s\}, P, S')$$

where P is:

$$\begin{aligned} S' &\to iCtS'eS' \mid iCtS \mid s \\ S &\to iCtS \mid s \\ C &\to c \end{aligned}$$

We claim that this is unambiguous as the grammer branches out from $iCtS'eS'$ and $iCtS$ and can't clash due to the presence of an extra e.

It is easy to notice that G' and G are equivalent. □