Gold Grabbing Games on Trees

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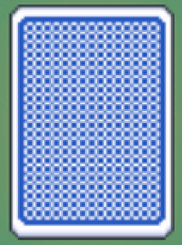
GRABBING CARDS!



Twenty-six cards laid between you. Take from either end, one at a time. The higher sum wins.

THE RULES

- 26 cards are dealt faceup in a line
- Players alternate picking from either end
- Card values: Ace = 1, Numbers = face, J = 11, Q = 12, K = 13
- Highest total when cards run out wins

DEAL THE CARDS

The absolute gorgeous cards were assets by https://kerenel.itch.io/pixelart-cards
This website uses the beautiful M6x11 font, a font by Daniel Linssen

Strategy!

So how does P1 keep winning?

Notice as the number of cards is even, Player 1 can choose to claim all the odd cards or all the even cards, and hence half the total of all the cards in play.

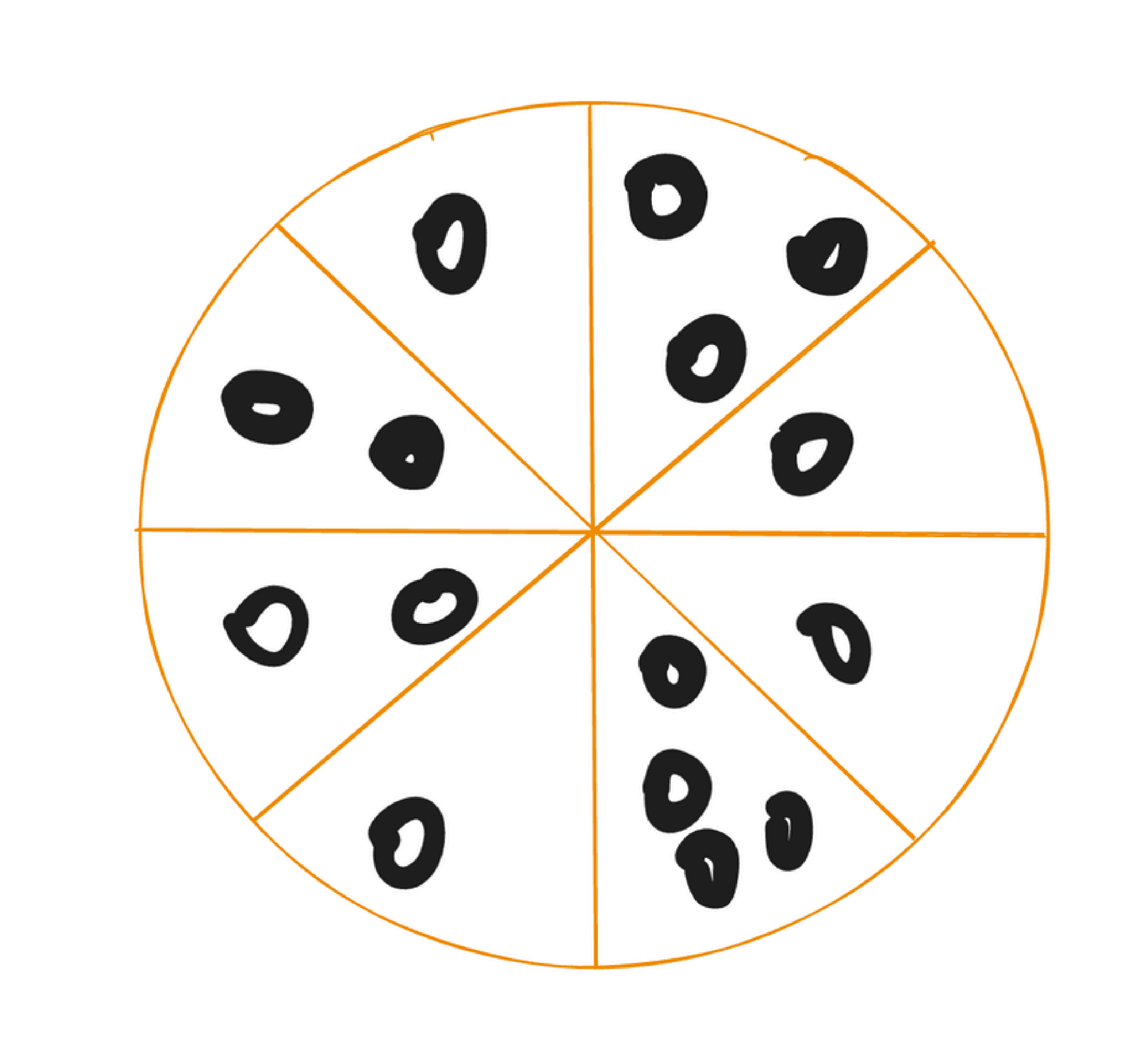
Notice, we don't claim that this is the optimal play that is we don't maximize the possible sum of our cards; we just beat our opponent.

As for finding the optimal strategy, we can use a simple $O(n^2)$ DP (and also an $O(n)$ greedy as we'll see).

A natural question is if we can generalize this?

Sharing Pizza

Let's talk about another similar game. Say we have a pizza with somewhat unevenly distributed toppings, say olives.



You want to share the pizza with a friend and both of you want to get more olives than each other.

For the sake of politeness, while P1 can take any slice on move 1; on subsequent turns only slices with atleast 1 neighbour missing can be taken.

We can see that a similar strategy would work here as well for Pizza's with even slices.

Graph Grabbing

We can generalize this game to general graphs.

Each vertex is assigned a score and a vertex can only be grabbed if its deletion doesn't increase the number of connected components of the graph.

This problem has been shown to be PSPACE complete. :-(

But there are 2 things we are interested in talking about:

- Graph Grabbing on Bipartite Graphs
- Graph Grabbing on Trees

Goals

It is conjuctured that

- There is polytime algorithm to determine the optimal strategy on a tree
- P1 wins on all bipartite graphs with an even number of vertices

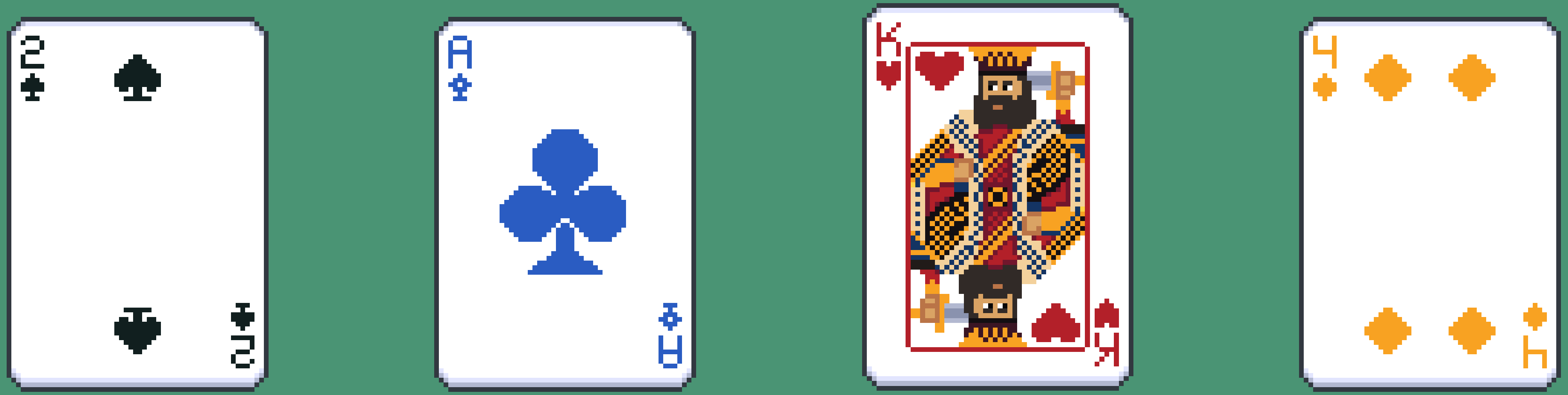
We'll share an idea that has recived little attention, first published by Tomasz Idziaszek of University of Warsaw as part of the editorial of Algorithmic Engagements 2010 and then in the beautiful book "Looking For a Challenge". It currently sits at a whopping 0 citations.

We feel these ideas could help resolve these conjuctures as well as simplify some of the existing proofs.

Grabbing Cards

Let's try to solve grabbing cards optimally in O(n) time.

Would a simple greedy strategy of simply taking the maximum of ends work? NO!



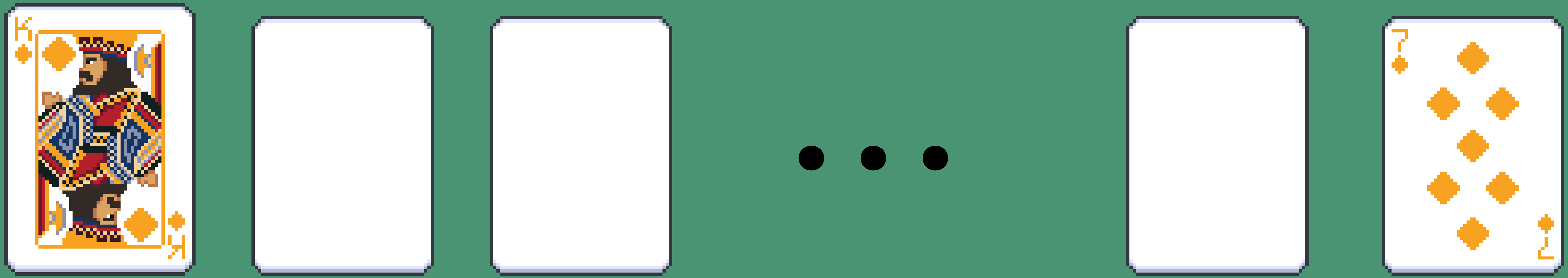
Are there some cases where this works out? Maybe when the maximum of the ends is also the maximum of the full set of card.

Observation 1: If one of the ends is the maximum of all cards, taking it is optimal

Observation 1

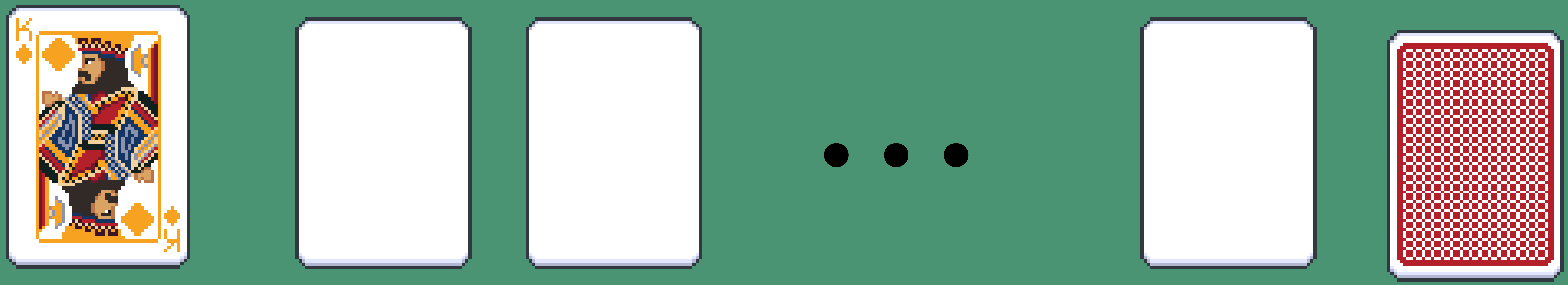
We can see this is clearly true if we have 1 or 2 cards.

Assume that this is true for n cards. We will show this is true for n+1 cards.

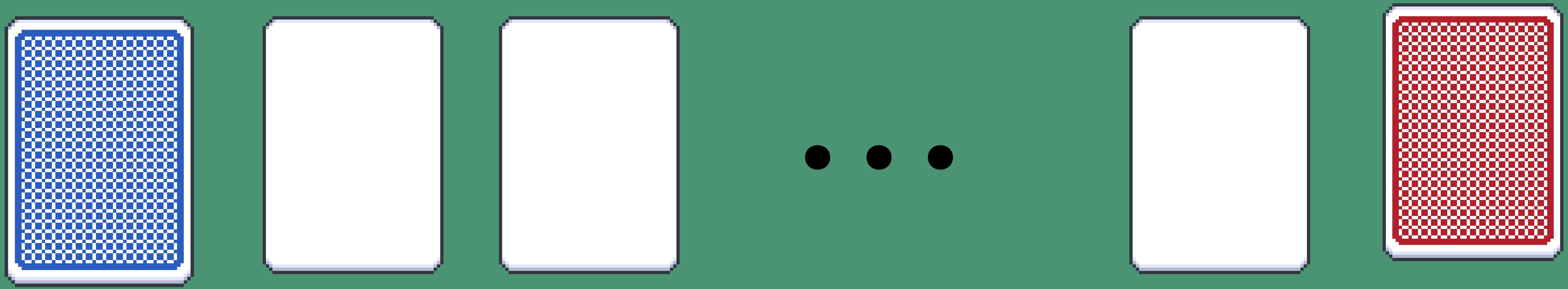


For the sake of contradiction, assume that taking K is not optimal. Let val(G) refer to the diffrence between scores of P1 and P2, given ideal play.

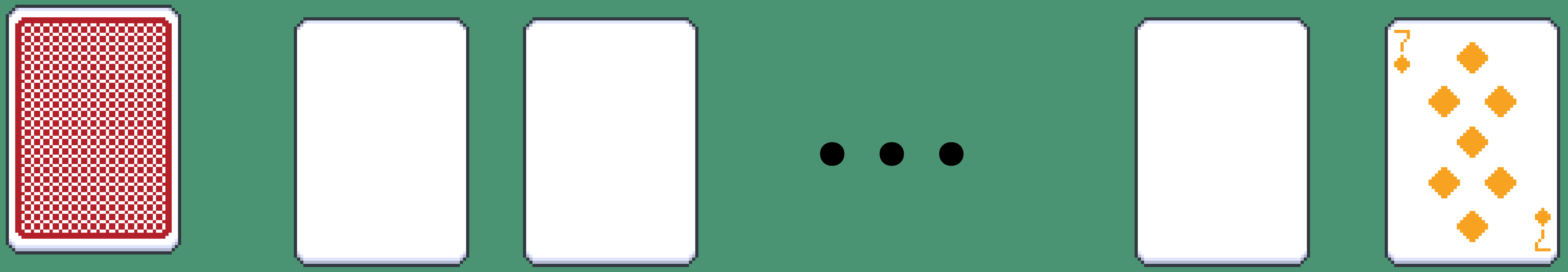
This implies P1 takes the other end.



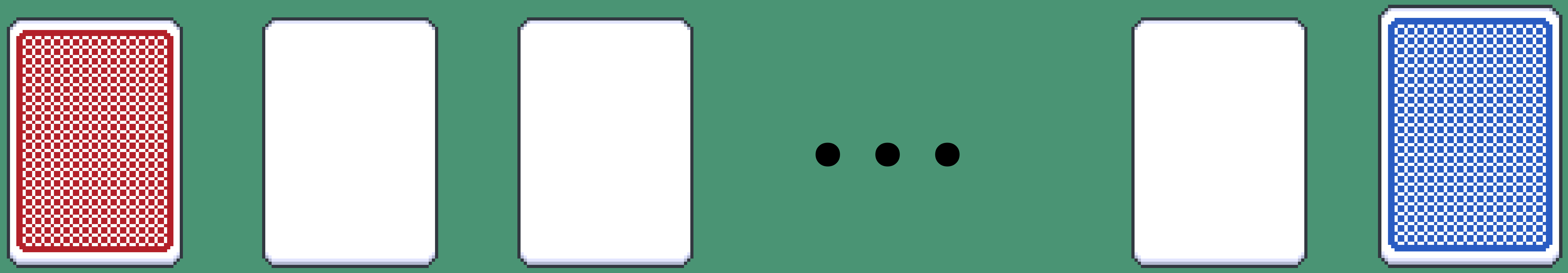
Then by the induction hypothesis, P2 takes the maximum element.



If P1 took the maximum element,



P2 has two choices. Let's assume taking the end P1 didn't is worse.



$$x - M + Val(G / x, M) \geqslant M - Val(G / M)$$
$$\geqslant M - x + Val(G/M, x)$$

But that is a contradiction as $M > x$.

Thus, our intial assumption must be false and hence, the induction hypothesis holds for n+1.

Thus, by induction, if one of the ends is the maximum of all cards, taking it is optimal.

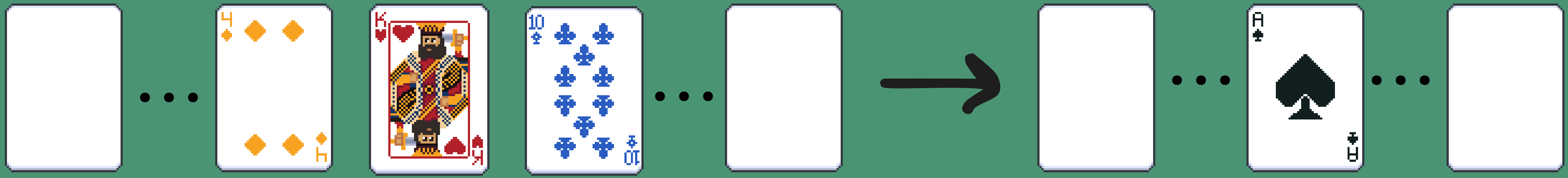
Let's name this the **Greedy Choice Principle**.

Observation 2

Unfortunately, we can’t use the Greedy Move Principal all the time!

Is their something else we could observe? Well, let’s say M is somewhere in the middle of the line. Then if a player makes a move uncovering it, it will be taken immidietly thanks to **Greedy Move.** But why did a player uncover it? Well, to probably take the thing just behind M.

This seems to suggest that we can replace xMy with x+y – M; and the value of the game won‘t change.



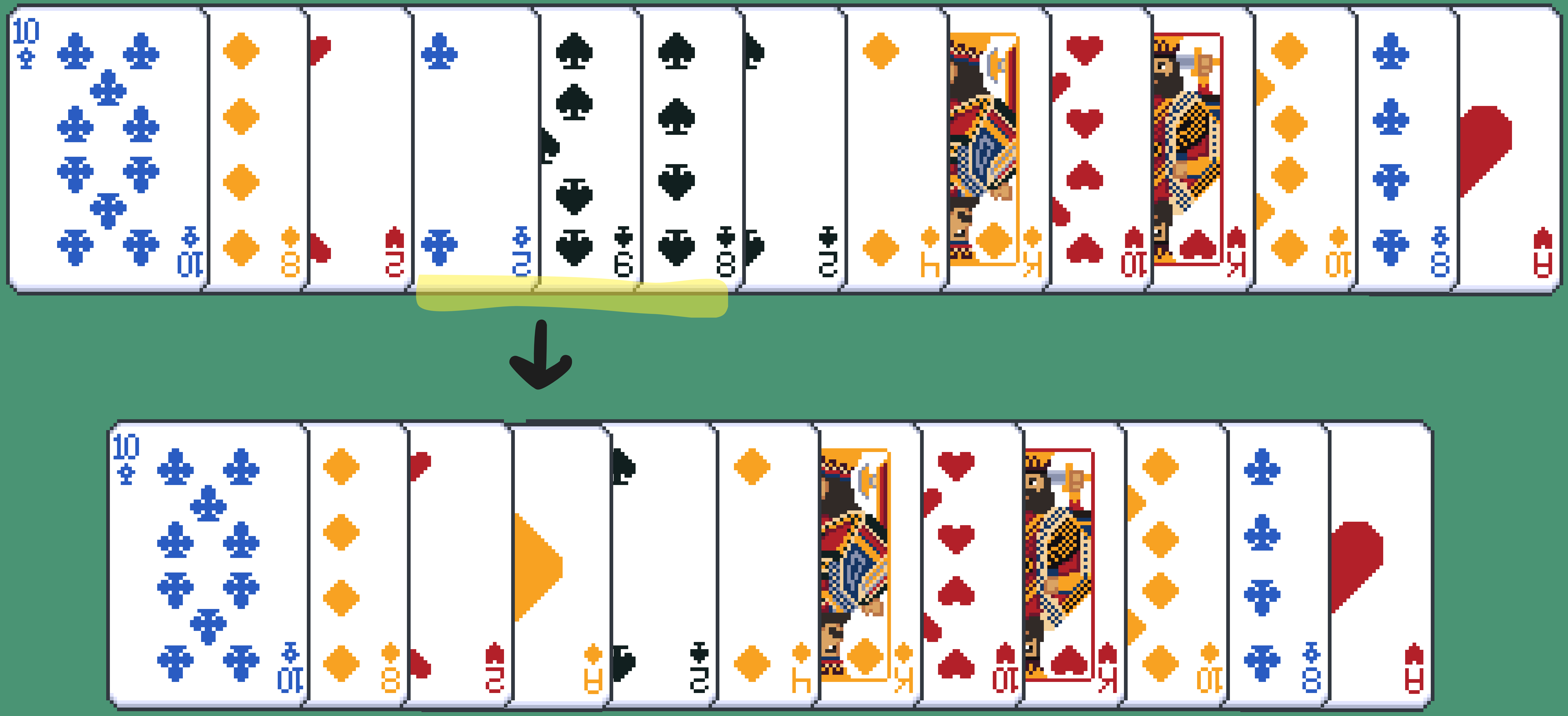
Fusion Principal

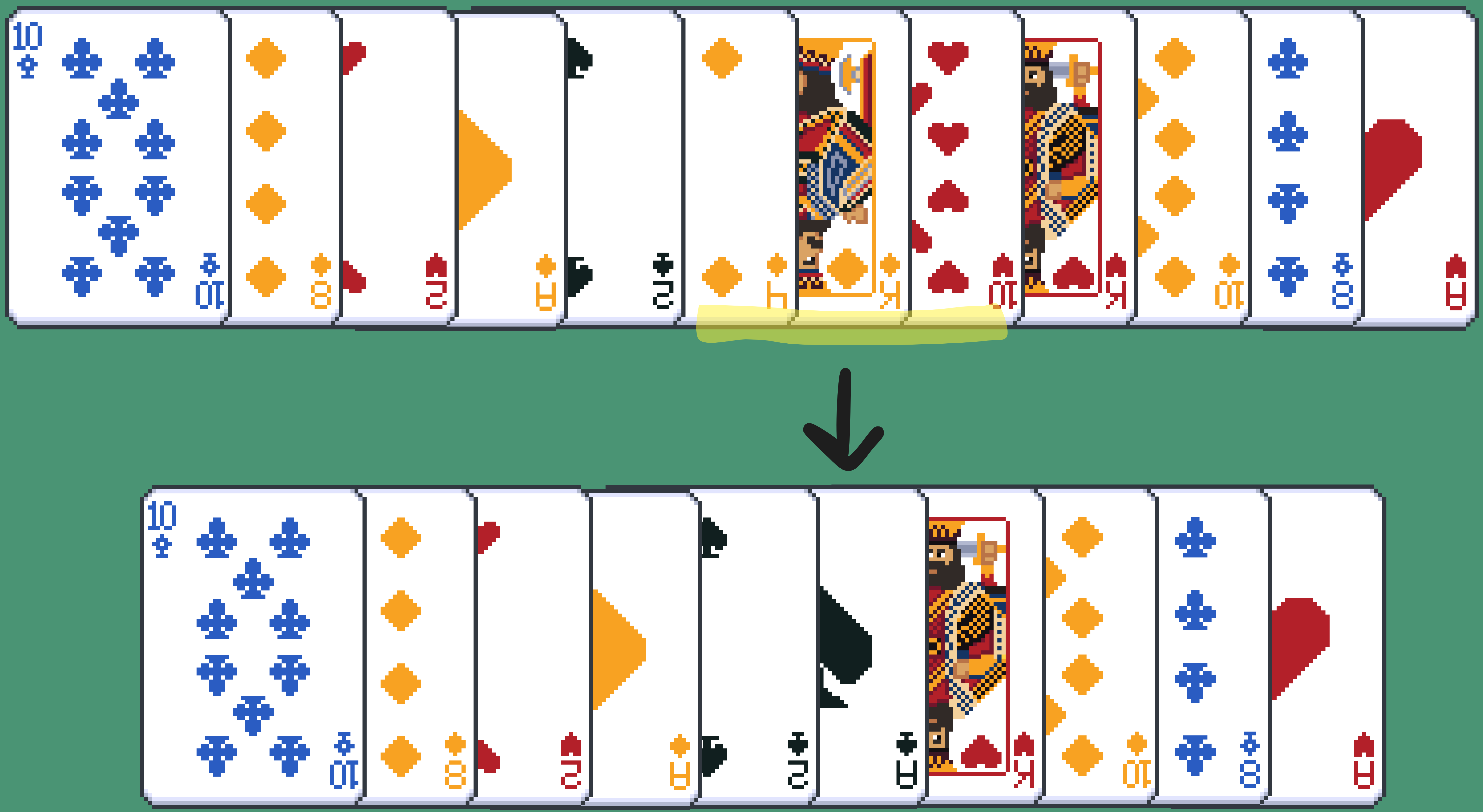
We will claim a more general statement:

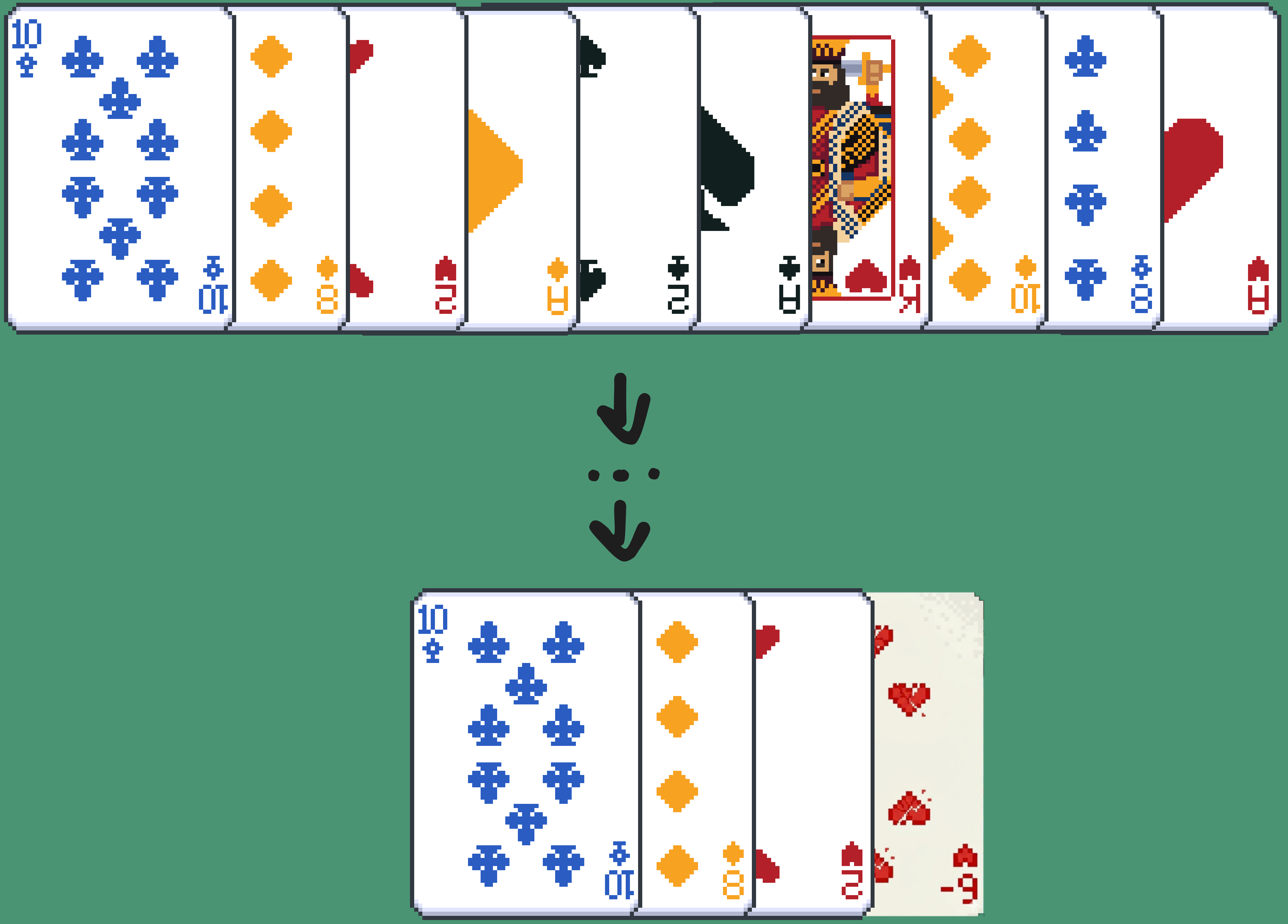
Fusion Principal: If x, M, y are adjacent cards such that $x, y \geqslant M$ then we can replace the three with a card of value $x - M + y$ and the value of the game will remain the same.

While this is a quite a strong statement, the proof is mostly book keeping and is skipped in favor of time.

And using these 2 observations, we have an $O(n)$ algorithm to find the optimal play.







The Algorithm

We mantain a stack and keep track of trios where we can apply the **fusion principal**.

If we can't use the fusion principal anymore, the sequence is bitonic (decreasing, then increasing). We just use **greedy principal** to compute the answer.

This clearly runs in O(n) time (and, if need be, in O(1) auxilary space!).

On Trees

Our proof for Greedy Move principal, without too many changes holds for trees.

Even the Fusion principal holds for any non-branching path of length atleast 3 in the tree.

However, this is not enough for trees. We will need a different Fusion Principal. Perhaps which can reduce 3 levels of the tree at the same time.

Similarly, there is a proof that for all trees with even vertices P1 can get atleast half the total but the proof doesn't give the strategy to do so.

Finally, we have not covered the idea the anchored graph grabbing games which are sometimes used in these proofs (and the **Worthless Move Principal**).

References

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That's All!

Any Questions?