Integral Bee Formula Sheet

$$\int x^n \, dx = \frac{x^{n+1}}{n+1}$$

$$\int \frac{1}{\sqrt{x}} \, dx = 2\sqrt{x}$$

$$\int \frac{1}{x} \, dx = \ln x$$

$$\int e^x \, dx = e^x$$

$$\int a^x \, dx = \frac{a^x}{\ln a}$$

$$\int \sin(x) \, dx = -\cos(x)$$

$$\int \cos(x) \, dx = \sin(x)$$

$$\int \sec^2(x) \, dx = \tan(x)$$

$$\int \csc^2(x) \, dx = -\cot(x)$$

$$\int \sec(x)\tan(x) \, dx = \sec(x)$$

$$\int \csc(x)\cot(x) \, dx = -\csc(x)$$

$$\int \frac{1}{\sqrt{1-x^2}} \, dx = \arcsin(x)$$

$$\int \frac{1}{1+x^2} \, dx = \arctan(x)$$

$$\int \frac{1}{x\sqrt{x^2-1}} \, dx = \sec^{-1}(x)$$

$$\int \tan(x) \, dx = -\ln|\cos(x)| = \ln\sec(x)$$

$$\int \cot(x) \, dx = \ln|\sin(x)| = -\ln\csc(x)$$

$$\int \frac{1}{a^2+x^2} \, dx = \frac{1}{a}\arctan\frac{x}{a}$$

$$\int \frac{1}{x^2-a^2} \, dx = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right|$$

$$\int \frac{1}{a^2-x^2} \, dx = \frac{1}{2a}\ln\left|\frac{x+a}{x-a}\right|$$

$$\int \frac{1}{\sqrt{x^2-a^2}} \, dx = \ln|x+\sqrt{x^2-a^2}|$$

$$\int \frac{1}{\sqrt{x^2+a^2}} \, dx = \ln|x+\sqrt{x^2+a^2}|$$

$$\int \sqrt{a^2-x^2}dx = \frac{1}{2}x\sqrt{a^2-x^2} + \frac{1}{2}a^2\arcsin\frac{x}{a}$$

$$\int \sqrt{a^2+x^2}dx = \frac{1}{2}x\sqrt{a^2+x^2} + \frac{1}{2}a^2\ln x + \sqrt{x^2+a^2}$$

$$\int \sqrt{x^2-a^2}dx = \frac{1}{2}x\sqrt{x^2-a^2} - \frac{1}{2}a^2\ln x + \sqrt{x^2-a^2}$$

$$\int \cos^2(\theta)d\theta = \frac{\theta}{2} + \frac{\sin(2x)}{4}$$

$$\int \sin^2(\theta)d\theta = \frac{\theta}{2} - \frac{\sin(2x)}{4}$$

Special Cases and Techniques

- For $\int \frac{1}{ax^2+bx+c}\,dx$ or $\int \frac{1}{\sqrt{ax^2+bx+c}}\,dx$, rewrite ax^2+bx+c as a perfect square and apply standard results.
- For $\int \frac{px+q}{ax^2+bx+c}\,dx$, express $px+q$ as the derivative of the denominator times a constant, then solve.
- If we wish to find $\int \frac{\cos(x)+\sin(x)}{f(\sin(2x))}dx$ we take $\cos(x)-\sin(x)=t$
- If we wish to find $\int \frac{\cos(x)-sin(x)}{f(\sin(2x))}dx$ we take $\cos(x)+\sin(x)=t$
- For $\int \frac{x^2\pm a}{x^4+kx^2+a^2}$ where k is a constant, divide the numerator and denominator by x^2 and then take $x \mp \frac{a}{x} = t$
- $\int \frac{1}{(px+q)\sqrt{ax+b}}dx; \int \frac{1}{(px^2+qx+r)\sqrt{ax+b}}dx$ Substitute $ax+b \to t^2$
- $\int \frac{1}{(px+1)\sqrt{ax^2+bx+c}}$ substitute $px+q=\frac{1}{t}$
- $\int \frac{1}{(px^2+q)\sqrt{ax^2+b}}$ substitute $x=\frac{1}{t}$
- For $\int \frac{1}{a\sin(x)+b\cos(x)+c}dx$, we will substitute $t=\tan(\frac{x}{2})$ and therefore $\sin x = \frac{2t}{1+t^2}$, $\cos x = \frac{1-t^2}{1+t^2}$, and $dx = \frac{2}{1+t^2}\,dt$
- For $\int \frac{1}{a\cos^2(x)+b\sin^2(x)+c\sin(x)\cos(x)}dx$ divide the numerator and denominator by $\cos^2(x)$ in order to take $\tan(x)=t$ and then solve.
- For $\int \frac{p\cos(x)+q\sin(x)+r}{a\cos(x)+b\sin(x)+c}dx$ we will try to express the numerator N as $N=\alpha D+\beta D'+\gamma$ where D is the denominator function.

Trigonometric Substitutions

- a^2-x^2 or $\sqrt{a^2-x^2}$: Substitute $x=a\sin(\theta)$ or $x=a\cos(\theta)$.
- a^2+x^2 or $\sqrt{a^2+x^2}$: Substitute $x=a\tan(\theta)$ or $x=a\cot(\theta)$.
- x^2-a^2 or $\sqrt{x^2-a^2}$: Substitute $x=a\sec(\theta)$ or $x=a\csc(\theta)$.
- $\sqrt{a+x}, \sqrt{a-x}, \sqrt{\frac{a+x}{a-x}}$ or $\sqrt{\frac{a-x}{a+x}} \to x = a\cos(2\theta)$
- $\sqrt{\frac{x-a}{b-x}}$ or $\sqrt{(x-a)(b-x)} \to x = a\cos^2(\theta) + b\sin^2(\theta)$

Integration By parts

$$\int udv = uv - \int vdu$$

$$\int e^{ax}\cos(bx)dx = \frac{e^{ax}}{a^2+b^2}(a\cos(bx)+b\sin(bx))$$

$$\int e^{ax}\sin(bx)dx = \frac{e^{ax}}{a^2+b^2}(a\sin(bx) - b\cos(bx))$$

$$\int e^x(f(x)+f'(x))dx = e^x f(x)$$

$$xf'(x)+f(x)dx = xf(x)$$

Cool Things

- King's Rule:

$$\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$$

- Feynman Trick:

$$\frac{d}{dt}\left(\int_a^b f(x,t)dx\right) = \int_a^b \frac{\partial}{\partial t}(f(x,t))dx$$

- Wallis' Integral:

$$\int_0^{\frac{\pi}{2}} \sin^m x\cos^n x\,\mathrm{d}x = \frac{(m-1)!!(n-1)!!}{(m+n)!!}k$$

- Gamma Function:

$$\Gamma(n) = \int_0^\infty x^{n-1}e^{-x}\,\mathrm{d}x$$

- The Beta function:

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

- The Riemann zeta function:

$$\zeta(s) = \sum_{n=1}^\infty \frac{1}{n^s} \text{ for } s>1$$

- Wallis' product:

$$\prod_{n=1}^\infty \frac{4n^2}{4n^2-1} = \frac{\pi}{2}$$

- For any integers $m \neq n$,

$$\int_0^{2\pi} \sin(mx)\sin(nx)dx = 0$$

$$\int_0^{2\pi} \cos(mx)\cos(nx)dx = 0$$

- For any integers m, n

$$\int_0^{2\pi} \sin(mx)\cos(nx)dx = 0$$

- Dirichlet's Integral:

$$\int_0^\infty \frac{\sin x}{x}dx = \frac{\pi}{2}$$

- Jail-breaking :

$$\int_a^b f(x)dx + \int_{f(a)}^{f(b)} f^{-1}(x) = bf(b) - af(a)$$

- Complexification:

$$\int \cos x f(x)dx = \Re(\int e^{ix}f(x))$$

$$\int \sin x f(x)dx = \Im(\int e^{ix}f(x)$$

- Frullani Formula:

$$\int_0^\infty \frac{f(ax)-f(bx)}{a} = (f(\infty)-f(0))\ln\left(\frac{a}{b}\right)$$

- Fresnel Integral:

$$\int_{-\infty}^\infty \cos(x^2)dx = \int_{-\infty}^\infty \sin(x^2)dx = \sqrt{\frac{\pi}{2}}$$